

FUNCTORIAL PROPERTIES OF THE SPACE $Z(X)$ FOR STRATIFIABLE SPACES

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ARTICLE INFO	ABSTRACT
Article History: Received 27 JUNE 2023 Accepted 9 OCTOBER 2023 Available online 15 DECEMBER 2023 Section Editor: Zainidin Eshkuvatov Keywords: Stratifiable space; simplex; dimension; category; covariant functor 2020 Mathematics Subject Classification: 54B15, 54B20, 54A25, 54B30, 54C15, 54C60, 54O30	In the realm of stratifiable spaces and their continuous mappings into themselves, a construction ϕ within the domain of Coty, known as the test space, defines a covariant functor within this category. This construction Z defines a functor $Z: S \rightarrow S$, which allows each stratifiable space X to be immersed in a closed manner into some other space $Z(X)$, which is a stratifiable space with “good” functorial, geometric and topological properties. It is shown that the functor $Z: S' \rightarrow S'$ is a normal, open and monadic functor in this category S of stratifiable spaces and continuous mappings into itself. Furthermore, an exploration of the dimensional properties of the space $Z(X)$ is conducted for the stratifiable space X, defined for each $n \in N_+$ subfunctor $Z_n: S \rightarrow S$ of the functor Z, for which the dimension satisfies the inequality: $\dim Z_n(X) \leq n \dim X + n - 1$.
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INTRODUCTION

A simplicial complex is defined as a set K for which its representation is fixed in the form of a union of some family of closed simplices, each two of which are not intersected, either one of them is a face of the other, or is intercepted by a simplex that is a face of each of them. The set K itself is called the body of the complex under consideration. In the presence of a complex, its body is uniquely determined, but one and the same set can be the body of various complexes. A simplicial complex is called complete if the complex K by an arbitrary given simplex $\sigma \in K$ also contains all the faces of this simplex.

Let K be a simplicial complex. By $K^{(n)}$, we denote the set of n -dimensional simplices (skeletons) of the simplicial complex K . $|K|$ is a polyhedron, i.e. $|K| = \cup K$ in weak topology.

For each simplex $\sigma \in K$, the barycentre, the interior and the boundary of the simplex σ are denoted by $\hat{\sigma}$, $\overset{\circ}{\sigma}$ and $\partial\sigma$, respectively. A simplex, whose vertices are $\vartheta_1, \vartheta_2, \dots, \vartheta_n$, is denoted by $\langle \vartheta_0, \vartheta_1, K, \vartheta_n \rangle$. Thus, $\tau \leq \sigma$ ($\tau < \sigma$) implies that τ is an integral part (facade or component) of the simplex

σ . For each point $x \in |K|$ through $(x(\tilde{\sigma}))$ of the simplex $\sigma \in K$, the barycentric coordinates with respect to the barycentre of the $S_d K$ subdivision are denoted.

A simplicial subdivision of a simplex $\sigma \in K$ is a family $P = \{\sigma_i; i = \overline{1, k}\}$ of simplices in K satisfying the following three conditions:

- a) $\sigma = \bigcup_{i=1}^k \sigma_i$;
- b) For any $i, j \leq k$, the intersection $S_i \cap S_j$ is either empty or is a common face of the simplices σ_i and σ_j .
- c) For $i = 1, 2, \dots, k$, all faces of the simplex σ_i belong to P .

The fineness of the simplicial subdivision $\{\sigma_i; i = \overline{1, k}\}$ of the simplex σ is defined as the largest among the numbers $\delta(\sigma_1), \delta(\sigma_2), \dots, \delta(\sigma_k)$. For each simplex $\sigma = \overline{a_0, a_1, \dots, a_m}$, the point

$$b(\sigma) = \frac{1}{m+1}a_0 + \frac{1}{m+1}a_1 + \dots + \frac{1}{m+1}a_m$$

is referred to as the barycentre (or centre of gravity) of the simplex σ . It is evident that $b(\sigma) \in \sigma$, and $b(\sigma)$ do not belong to any $(m-1)$ -dimensional face of the simplex σ .

Consider an arbitrary simplex $\sigma = \langle a_0, a_1, \dots, a_m \rangle$. For each decreasing sequence $\sigma_0 \supset \sigma_1 \supset \dots \supset \sigma_k$ of different faces of the simplex σ , the points $b(\sigma_0), b(\sigma_1), \dots, b(\sigma_k)$ are linearly independent. The family Π of all simplices of the form $\langle b(\sigma_0), b(\sigma_1), \dots, b(\sigma_k) \rangle$ will be a simplicial subdivision of the simplex σ . Each $(m-1)$ -dimensional simplex $T \in \Pi$ is a face of one or two m -dimensional simplices of the family Π , depending on whether the simplex is contained or not T in some $(m-1)$ -dimensional face of the simplex σ . This particular simplicial subdivision, as described above, is referred to as the barycentric subdivision of the simplex σ .

Consider a number $t \in [0, 1]$ and $\sigma(t) = \{x \in \sigma: 0 \leq \hat{x}(\sigma) \leq t, (\hat{x}(\sigma)) \text{ are barycentric coordinates of the point } x\}$. For each $t \in [0, 1]$, the set $\sigma(t)$ is an open neighbourhood of the boundary $\partial\sigma$ of the simplex σ , and $\sigma(t) = cl_\sigma \sigma(t)$. Note that $\sigma(1) = \sigma \setminus \{\hat{\sigma}\}$.

Each point $x \in \sigma(1)$ can be expressed as as:

$$x = (1 - x(\hat{\sigma})) \pi_\sigma(x) + x(\hat{\sigma}) \hat{\sigma},$$

where $\pi_\sigma(x) \in \partial\sigma$ and $\pi_\sigma(x): \sigma(1) \rightarrow \partial\sigma$ is a radial projection for each $\sigma \in K$. Furthermore, each point $x \in v_0, v_1, \dots, v_n$ can be written as:

$$x = \sum_{i=0}^n b(v_i) v_i,$$

where $v_i \in K^0$, and $b(v_i)$ represent the barycentric coordinates of the point x .

MAIN PART

Let X be a topological space, and $|F(X)|$ be a complete simplicial complex whose vertices are points of X , i.e. $|F(X)|^0 = X$. The space $|F(X)|$ is equipped with the weak topology. Now, a topology on the space $|F(X)|$ is defined, denoting the basis of open sets as $Z(X)$, which consists of W open in $F(X)$, and satisfy the following conditions:

- o1. $W \cap X$ open in X ;
- o2. $|F(W \cap X)| \subset W$;
- i.e. $\tau_{Z(X)} = \{W \in \tau_{|F(X)|}: W \text{ satisfies the conditions o1-o2}\}.$

Condition o2 means that every simplex $\sigma \in F(X)$ is contained in W if all vertices of this simplex σ lie entirely in $W \cap X$.

For the subset $A \subset X$, the set $F(A)$ is a subcomplex of the full complex $F(X)$ and $Z(X)$ is a subspace of the space $Z(X)$. Obviously, $Z(A)$ is closed in $Z(X)$ if A is closed in X .

For each $n \in Z_+ = N \cup \{0\}$, define $Z_n(X) = |F(X)^n|$, which is a subspace of $Z(X)$. Then, $Z_0(X) \cong X$ and $Z(X) = \bigcup_{n=0}^{\infty} Z_n(X)$. It is evident that for any $n \in Z_+$, the subspace $Z_n(X)$ is closed in $Z(X)$.

Let us introduce the following notation:

$$T(A) = \{\sigma \in F(X) \setminus F(A) : \sigma \cap A \neq \emptyset\};$$

$$M(A) = \{x \in Z(X) : \text{there exists } \sigma \in F(A) \text{ such that } x(\sigma) < 0\};$$

$$T_n(A) = T(A) \cap (F(X)^n \setminus F(X)^{n-1});$$

$$M_n(A) = Z(A) \cup (M(A) \cap Z_n(X));$$

For each $\varepsilon \in (0, 1)^{T(A)}$ and for each $n \in N$, we define the set:

$$M(A, \varepsilon) = \bigcup_{n \in Z_+} M_n(A, \varepsilon),$$

where $M_0(A, \varepsilon) = Z(A) = |F(A)|$ and

$$M_n(A, \varepsilon) = Z(A) \cup \{\sigma(\varepsilon(\sigma)) \cap \pi_{\sigma}^{-1}(M_{n-1}(A, \varepsilon)) : \sigma \in T_n(A)\}.$$

Then the equality $M_n(A, \varepsilon) \cap X = A$ holds.

For each open set U of the space X , the set $M(U, \varepsilon)$ is open in $Z(X)$. In this case, the family $B(M) = \{(U, \varepsilon) : U \text{ is open in } X \text{ and } \varepsilon \in (0, 1)^{(U)}\}$ is an open base of the space $Z(X)$.

Therefore, if for every $n \in N$ and every $\varepsilon \in (0, 1)^{T_1(A) \cup T_2(A) \cup \dots \cup T_n(A)}$, the set $M_n(A, \varepsilon)$ is defined, then the family $B(M) = \{(M_1, (U, \varepsilon)) : U \text{ is open in } X \text{ and } \varepsilon \in (0, 1)^{T_1(U)}\}$ is an open base for $Z_1(X)$, i.e. the following holds.

Lemma [5]. Families $\{M(U, \varepsilon) : U \text{ is open in } X \text{ and } \varepsilon \in (0, 1)^{T_1(U)}\}$ and $\{(M_1, (U, \varepsilon)) : U \text{ is open in } X \text{ and } \varepsilon \in (0, 1)^{T_1(U)}\}$ is the base of the space $Z(X)$, (respectively, the space $Z_1(X)$).

Let $A \subset X$, then the subspace $Z(X) \setminus Z(A)$ of the space $Z(X)$ consists of points of the form: $z = (1-t)x + ty$, where $x \in Z(X \setminus A)$, $y \in Z(A)$, $t \in [0, 1)$, i.e.

$$Z(X) \setminus Z(A) = \{z = (1-t)x + ty \in Z(X) : x \in Z(X \setminus A), y \in Z(A), t \in [0, 1)\}.$$

Obviously, if the set A is closed in X , then $Z(A)$ is closed in $Z(X)$. It should be noted that the set itself is also closed in $Z(X)$, i.e. $X; Z_1(X)$.

Note that for $z \in Z(X)$, based on the structure of the space $Z(X)$, the point has the form: $z = m_0 \bar{x}_0 + m_2 \bar{x}_2 + \dots + m_n \bar{x}_n$, where $\sum_{i=0}^n m_i = 1$, $m_i \geq 0$, and $x_i \in X$. In other words, the point z belongs to the simplex $\sigma = \langle \bar{x}_0, \dots, \bar{x}_n \rangle$, whose vertices are points x_i of space X , denoted by $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_n$, which are the points (vertices) of the simplex $\sigma = \langle \bar{x}_0, \dots, \bar{x}_n \rangle$.

If $z \in Z_1(X)$, then the point z has the form $z = x$, where $x \in X$. Hence, for each point $z \in Z(X) \setminus Z_1(X)$ there exists a simplex $\sigma \in Z(X)$ such that $z \in \sigma = \langle \bar{x}_0, \bar{x}_1, \dots, \bar{x}_n \rangle$ and $z = \sum_{i=0}^n m_i \bar{x}_i$, $m_i > 0$, and $\sum_{i=0}^n m_i = 1$. The vertices of the simplex $\sigma = \langle \bar{x}_0, \bar{x}_1, \dots, \bar{x}_n \rangle$ are referred to supports of the point $z \in \sigma$, i.e. $\text{supp } z = \{x_0, x_1, \dots, x_n\}$.

In this case, there exists at least j_1 and $j_2 \in \overline{0, n}$ such that $m_{j_1} \neq 0$ and $m_{j_2} \neq 0$, indicating that the point $z \in Z(X) \setminus Z_1(X)$ has at least two points in its support. Therefore, if $z \in Z(X) \setminus Z_1(X)$, then $|\text{supp} z| \geq 2$. On the other hand, given the construction of the space $Z(X)$, for any $n \in \mathbb{N}$, there is a point $z \in Z(X)$ such that $|\text{supp} z| \geq n$. If $z \in Z_n(X)$, then $|\text{supp} z| \geq n$.

In the paper [1] by R. Coty, it was demonstrated that the space $Z(X)$ is stratifiable (in short, a S -space) if and only if X is a S -space.

Let $f: X \rightarrow Y$ be a continuous mapping between the S -spaces X and Y . The continuous mapping $Z(f): Z(X) \rightarrow Z(Y)$ is defined as follows: for each point $z \in Z(X)$ and $z \in \langle \bar{x}_0, \dots, \bar{x}_k \rangle$

$z_x = m_0 \bar{x}_0 + \dots + m_k \bar{x}_k$, $\sum_{i=0}^k m_i = 1$, $m_i \geq 0$ assign point $z_y \in Z(Y)$, $z_y = \langle \bar{y}_0, \dots, \bar{y}_k \rangle$ $z_y = m_0 \bar{y}_0 + m_1 \bar{y}_1 + \dots + m_k \bar{y}_k$ where $f(\bar{x}_i) = \bar{y}_i$.

The mapping $Z(f): Z(X) \rightarrow Z(Y)$ is simplicial. Therefore, $Z(f)$ is continuous. Note that $Z(f)(Z(X)) = Z(f(X))$ and $Z(f)(Z(X) \setminus X) \subset Z(Y) \setminus Y$. Hence, if the mapping $f: X \rightarrow Y$ is surjective, then $Z(f): Z(X) \rightarrow Z(Y)$ is also surjective and vice versa.

- If $id_X: X \rightarrow X$ is the identity mapping, then $Z(id_X): Z(X) \rightarrow Z(X)$ is also the identity mapping, i.e. $Z(id_X) = id_{Z(X)}$.
- Let $p: X \rightarrow Y$ and $g: Y \rightarrow L$ be continuous mappings between the stratifiable spaces X, Y and L .

If $f = g \circ p: X \rightarrow L$, where the diagram

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ & f \downarrow g & \\ & L & \end{array} \quad (1)$$

is commutative, then $Z(f) = Z(p \circ g): Z(X) \rightarrow Z(L)$, which is the diagram

$$\begin{array}{ccc} Z(X) & \xrightarrow{Z(p)} & Z(Y) \\ & Z(f) \downarrow Z(g) & \\ & Z(L) & \end{array} \quad (2)$$

is also commutative. Hence, we see that the mapping $Z: X \rightarrow Z(X)$ forms a covariant functor between stratifiable spaces and their continuous mappings.

If $f = p \circ g$, then $Z(f) = Z(p \circ g) = Z(p) \circ Z(g)$. Hence, the mapping of the stratifiable space $Z: X \rightarrow Z(X)$ is functorial. Each mapping $f: X \rightarrow Y$ can be associated with the above defined simplicial mapping $Z(f): Z(X) \rightarrow Z(Y)$. From properties a) and b), we can assert that this mapping Z is a covariant functor on the category of S -spaces and continuous mappings into itself.

Thus, the following is true:

Theorem 1. *is a covariant functor.*

Note that the functor $Z: S \rightarrow S$ has the following properties:

- From the construction of the space $Z(X)$ and Lemma [5], it follows that the functor $Z: S \rightarrow S$ preserves the entire space $\omega(Z(X)) \leq \omega(X)$, i.e. $\omega(Z(X)) \leq \omega(X)$.
- For $X \in S$, the mapping or embedding $\eta_X: X \rightarrow Z(X)$ is constructed by defining $\eta(X) = \langle x \rangle$, where $x \in Z(X)$, and x is vertex, i.e. x is a zero-dimensional simplex in $Z(X)$. The mapping $\eta: X \rightarrow Z(X)$ is continuous and one-to-one, implying that $\eta_X: X \rightarrow Z(X)$ is an embedding.

An immediate check shows that the functor $Z: S \rightarrow S$ preserves

- 3°. a point and an empty set;
- 4°. intersection of closed sets;
- 5°. preimages of mappings;
- 6°. continuous in the sense of Shchepin [2];
- 7°. monomorphic; and,
- 8°. epimorphic.

Note that the functor $Z: S \rightarrow S$ with finite supports does not have finite degrees. Hence, the functor $Z: S \rightarrow S$ preserves a point, an empty set, intersections of closed sets, and inverse images of mappings. It is continuous, epimorphic, and monomorphic, i.e. the functor $Z: S \rightarrow S$ is normal.

For each $n \in N_+$ define $Z_n(X) = \{z \in Z(X) : |\text{supp} z| \leq n-1\}$, where $z_x = \sum_{i=0}^n m_i \bar{x}_i$, $\text{supp} z_x = \{x_0, x_1, \dots, x_n\}$, $\sum_{i=0}^n m_i = 1$, $m_i \geq 0$, $\text{supp} z_x \subset X$.

- a) The set $Z_n(X)$ is closed in $Z(X)$;
- b) for $n = 0$, $Z_0(X) \cong X$;
- c) for each mapping $f: X \rightarrow Y$ between S spaces, the mapping $Z_n(f): Z_n(X) \rightarrow Z_n(Y)$ is defined as follows: $Z_n(f) = Z(f)|_{Z_n(X)}: Z_n(X) \rightarrow Z_n(Y)$.

The mapping $Z_n(f): Z_n(X) \rightarrow Z_n(Y)$ is also continuous, i.e. $Z_n: S \rightarrow S$ is a subfunctor of the functor Z . The functor $Z_n: S \rightarrow S$ possesses all properties of the functor $Z: S \rightarrow S$. Functor $Z_n: S \rightarrow S$ is a functor with finite n -supports.

Since the mappings $Z(f)$ and $Z_n(f)$ are simplicial, we can assert:

Theorem 2. *A mapping $f: X \rightarrow Y$ between S -spaces is open if and only if $Z(f): Z(X) \rightarrow Z(Y)$ is open (or $Z_n(f): Z_n(X) \rightarrow Z_n(Y)$ is open).*

The following is easily proved.

Theorem 3. *A map $f: X \rightarrow Y$ between S -spaces mapping $Z(f): Z(X) \rightarrow Z(Y)$, $Z_n(f): Z_n(X) \rightarrow Z_n(Y)$ is a homeomorphism if and only if f is a homeomorphism.*

Note that the covariant functors $Z: S \rightarrow S$ and $Z_n: S \rightarrow S$ is a normal functor.

A monad (or triplet) on the category G is a triplet $T = \langle F, \psi, \eta \rangle$, where $F: G \rightarrow G$ is a functor $\eta: Id \rightarrow F$ (unit) and $\psi: F^2 \rightarrow F$ (multiplication) are natural transformations. Additionally, for each object X , the following equalities are satisfied:

$$\psi_X \circ F(\eta_X) = id_{F(X)},$$

$$\psi_X \circ \eta_{F(X)} = id_{F(X)}$$

$$\psi_X \circ F(\psi_X) = \psi_X \circ \psi_{F(X)}$$

A functor F is called monadic if it can be included in some monad.

Theorem 4. *The covariant functor $Z: S \rightarrow S$ is monadic.*

Proof. As previously mentioned, the mapping $\eta: Z(X) \rightarrow Z(Y)$ which assigns to each point $x \in X$ the vertex $\bar{x}$ of the simplex in $Z(X)$ is continuous, i.e. $\eta(x) = \bar{x}$.

Let $z \in Z(Z(X)) = Z^2(X)$. Then the point z has the form $z = \sum_{k=1}^k \mu_i z_i$, where $z_i \in Z(X)$,

$$z_i = \sum_{j=0}^{l_i} m_s^i \overline{x_s^j}, \sum_{i=0}^k \mu_i = 1, \sum_{s=0}^{l_i} m_s^i = 1, x_s^j \in X,$$

$$z = \sum_{i=0}^k \mu_i z_i = \mu_0 z_0 + \mu_1 z_1 + \dots + \mu_k z_k = \mu_0 \sum_{s=0}^{l_0} m_s^0 \overline{x_s^0} + \mu_1 \sum_{s=0}^{l_1} m_s^1 \overline{x_s^1} + \dots + \mu_k \sum_{s=1}^{l_k} m_s^k \overline{x_s^k} = \sum_{i=0}^k \sum_{s=0}^{l_i} \mu_i m_s^i \overline{x_s^i}; \text{ We}$$

put $a_s^i = m_s^i \mu_i \geq 0$, $\sum_{i=0}^k \sum_{s=0}^{l_i} a_s^i = 1$. Hence, $z = \sum_{i=0}^k \sum_{s=0}^{l_i} a_s^i \overline{x_s^i}$, $\overline{x_s^i} \in X$. Hence, $z \in Z(X)$ that is the mapping $\psi \in Z(Z(X)) \rightarrow Z(X)$ is defined. The mapping Ψ_x is continuous because Ψ_x is linear.

A simple check shows that for the triple $Z = Z, \Psi, \eta_X$ satisfies the following conditions:

- (a) $\Psi_X \circ Z(\eta_X) = id_{Z(X)}$;
- (b) $\Psi_X \circ \eta_{Z(X)} = id_{Z(X)}$;
- (c) $\Psi_X \circ Z(\Psi_X) = \Psi_X \Psi_{Z(X)}$;

$Z: S \rightarrow S$ functor, $\eta_Z: id \rightarrow Z$ (one) and $\Psi: Z^2 \rightarrow Z$ (multiplication) natural transformation. Hence, the triple $T = \langle Z, \Psi, \eta_X \rangle$ is monadic. Theorem 4 is proved.

Theorem 5. For any stratifiable space X , the space $Z(X)$ is convex and the subspace $Z_v(X)$ is homotopically dense in $Z(X)$.

Proof. Let $X \in S$. Two cases are possible : a) X is finite. b) X is infinite. In case a) the space $Z(X)$ consists of one simplex $\sigma = \langle x_0, x_1, \dots, x_n \rangle$. This simplex σ is convex and the subspace $Z_v(X) = Z(X) \setminus X = \sigma \setminus \{x_0, x_1, \dots, x_n\}$ is homotopy and dense in $Z(X)$. Let X be infinite. In this case, the space $Z(X)$ is convex. Indeed, for any points $z_0 \in Z(X)$, $z_1 \in Z(X)$, the segment $[z_0, z_1]$ lies entirely in the space $Z(X)$, i.e. $[z_0, z_1] \subset Z(X)$. Now we can that the subspace $Z_v(X)$ is homotopically dense in $Z(X)$.

We construct the desired homotopy $h(z, t): Z(X) \times [0, 1] \rightarrow Z(X)$ supposing $h(z, t) = (1-t)z + t \cdot z_0$, where $z \in Z(X)$, $t \in [0, 1]$, $z_0 = m_0 \bar{x}_0 + \dots + m_n \bar{x}_n$, $\sum_{i=1}^n m_i = 1$, $m_i > 0$, $m_i \neq 0$.

If $t = 0$, then $h(z, 0) = (1-0)z + 0 \cdot z_0 = z$, i.e. $h(z, 0) = id_{Z(X)}$.

If $t \in (0, 1]$, that is, $t > 0$, $t \leq 1$, then the point $h(z, 1) = (1-t)z + t \cdot z_0 \in Z_1(X)$, i.e. $h(z, 0) = id_{Z(X)}$, i.e. point $h(z, 1) \in Z_v(X)$. This means that the subspace $Z_v(X)$ is homotopically dense in $Z(X)$. The theorem is proved.

A subset $A \subset X$ of the space X is said to be homotopically dense in X [9] if there exists a homotopy $h: X \times [0, 1] \rightarrow X$ such that $h(x, 0) = id_x$ and $h(X \times (0, 1]) \subset A$. Therefore, the subset $A \subset X$ is called homotopy negligible [9] if the complements $X \setminus A$ is homotopy dense in X . The above theorem implies.

Corollary. For any stratifiable space X , the space $Z_1(X)$ itself is homotopically negligible in $Z(X)$

Theorem 6. Let X be a finite-dimensional St -space. Then $Z_n(X)$ is a finite-dimensional St -space. Moreover, the inequality $\dim Z_n(X) \leq \dim X^n + n-1 \leq n \dim X + n-1$ is true.

Proof. Let X be a St -space and $\dim X < \infty$. Note that for any $n \in \mathbb{N}$ the space $Z_n(X)$ is also St -a space, therefore, $Z_n(X)$ is a perfectly normal paracompact space. It follows from the results of [8, 10] that

$$\dim X^n \times \sigma^{n-1} \leq \dim X^n + \dim \sigma^{n-1} \leq n \dim X + n - 1$$

Through induction on n , =

$$\dim Z_n(X) \leq n \dim X + n - 1 \quad (3)$$

For $n = 1$, this inequality (3) is valid, since the space $Z_1(X)$ is homeomorphic to X by definition. those $Z_1(X); X$.

Let the inequality $\dim Z_{n-1}(X) \leq (n-1) \dim X + \dim Z_{n-1}(\overline{n-1})$. The theorem is proved.

First, it is demonstrated that $rd_{Z_n(X) \setminus Z_{n-1}(X)} \leq n \dim X + n - 1$, where $rd_X A$ is the relative dimension of the subspace A in the space $[7]$. The subspace $Z_n(X) \setminus Z_{n-1}(X)$ is also St -space, hence it is paracompact. The space $Z_n(X)$ at the points of the set $Z_n(X) \setminus Z_{n-1}(X)$ is locally homeomorphic to the open subspace of the product $X^n \times \sigma^{n-1}$. From Theorem 21 of [7] Dowker-Nagami we obtain that $rd_{Z_n(X) \setminus Z_{n-1}(X)} \leq n \dim X + n - 1$.

Then, applying another Dowker's theorem 10 [7], we obtain that

$$\dim Z_n(X) \leq n \cdot \dim X + n - 1.$$

The theorem 6 is proved.

A space X is weakly star countable if for each open cover U of X there exists a countable subset F of X such that $(\bigcup_{x \in F} St(x, U)) = X$.

Theorem 7. *If X is a weakly countable St -space, then $Z_n(X)$ is also weakly countable.*

Theorem 8. *For any weakly countable or finite-dimensional St -space, the space $Z(X)$ is weakly countable.*

Theorem 9. *For any infinite St -space X , the space $Z(X)$ is infinite-dimensional.*

Definition. For the subset of the normal space X , we write $[7] rd_X A \leq n$ (relative dimension A in $X \leq n$), if for any set F lying in A and closed in X the inequality $\dim F \leq n$, $n = -1, 0, 1, 2, \dots$.

Theorem 10 [7]. *If in the normal space X there exists a closed set F such that $\dim F \leq n$ and $rd_X (X \setminus F) \leq n$, then $\dim X \leq n$.*

Theorem 11 [7]. *For paracompact the following is always true*

$$\text{loc dim } X = \dim X$$

Definition [7]. For a normal space X , we assume $\text{loc dim } X \leq n$ if for any point $x \in X$ there exists a neighborhood Ox for which $\dim[Ox] \leq n$, $n = 0, 1, 2, \dots$. (For empty X we get $\text{loc dim } X = -1$).

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

REFERENCES

- [1] Borges, C. R. (1996). On stratifiable spaces. *Pacific Journal of Mathematics*, 17(1), 1-16.
- [2] Robert, C. (1972). Retractions dans les espaces stratifiables. *Bulletin de la Société Mathématique de France*, 102, 129-149. <https://eudml.org/doc/87222>
- [3] Borges, C. R. (1969). A study of absolute extensor spaces. *Pacific Journal of Mathematics*, 31(3), 609-617. <https://msp.org/pjm/1969/31-3/pjm-v31-n3-p07-s.pdf>
- [4] Borsuk, K. (1971). *The theory of retracts* (pp. 291). Moscow: Mir Publisher.
- [5] Robert, C., Bao-Lin, G., & Sakai, K. (1995). The hyperspace of finite subsets of a stratifiable space. *Fundamenta Mathematicae*, 147, 1-9. <https://bibliotekanauki.pl/articles/1208364.pdf>
- [6] Zhuraev, T. F. (2014). *Equivariant analogs of some geometric and topological properties on stratifiable spaces X* (pp. 23-27). West. Kirg. Nat. University.
- [7] Aleksandrov, P. S., & Pasynkov, B. A. (1973). *Introduction to the theory of dimension* (pp. 576). Moscow, Izdatel'stvo Nauka.
- [8] Zhuraev, T. F. (1989). *Some geometric properties of the functor of probabilistic measures and its subfunctors* [Doctoral's Dissertation Thesis, MGU, Moscow].
- [9] Banakh, T., Radul, T., & Zarichniy, M. (1996). *Absorbing sets in infinite-dimensional manifolds*. (Mathematical Studies Monograph Series, Vol. 1). VNTL Publishers.
- [10] Zhuraev, T. F. (1992). σ -spaces and functors of finite degree. *Reports of Uzbekistan Academy of Sciences*, 4-5, 15-18.