Universiti Malaysia Terengganu Journal of Undergraduate Research
Volume 7 Number 1, April 2025 39-44
eISSN: 2637-1138

LINDLEY'S APPROXIMATION IN ESTIMATING TWO-PARAMETERS INVERTED WEIGHTED EXPONENTIAL DISTRIBUTION

FATIN SYAZWANI NAZRI[1,2] AND HANI SYAHIDA ZULKAFLI[1]*

[1]*Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.* [2]*Department of Informatics, Graduate School of Integrated Science and Technology, Shizuoka University, Shizuoka Prefecture, Japan.*

**Corresponding author: hsyahida@upm.edu.my*
https://doi.org/10.46754/umtjur.v7i1.450
Received: 14 September 2023 *Revised: 29 July 2024* *Accepted: 4 September 2024* *Published: 15 April 2025*

Abstract: The Inverted Weighted Exponential (IWE) distribution can be used to describe and model real-life phenomena with unimodal or decreasing failure rates. The main aim of this study is to use the Bayesian approach to estimate the two parameters of the distribution. Two loss functions are used, the Squared Error Loss Function (SELF) and Linear Exponential (LINEX) loss function, using Lindley's method. The Root Mean Square Error (RMSE) is reviewed to assess the performance of the estimation method. The results of simulation studies reveal that the scale parameter benefitted from using the Bayesian approach with a SELF. This paper mainly aims to demonstrate how the Bayesian framework can be used to estimate the parameters of the IWE distribution in addition to the conventional Maximum Likelihood estimation (MLE). A detailed simulation study is conducted to compare the performance of the Bayesian estimator and the non-Bayesian estimator, i.e., MLE. The Bayesian approach uses Lindley's approximation with SELF and LINEX functions. The empirical results demonstrate that the scale parameter of the IWE distribution benefitted from using the Bayesian approach with a SELF. The estimators also perform better with a larger sample size, n. Nevertheless, no similar work has used Bayesian estimation for the parameters of the IWE distribution.

Keywords: Bayesian estimation, linear exponential loss function, squared error loss function, maximum likelihood estimation.

Introduction

Weighted Exponential (WE) distribution with two parameters was developed by Gupta and Kundu (2009) and are being used as a lifetime model in the medical and engineering fields. An inverted version of the WE, named Inverted Weighted Exponential (IWE), was then proposed (Oguntunde *et al*., 2018) and revealed to perform better when applied to real-life data (Bhaumik *et al*., 2009; Oguntunde *et al*., 2017) compared to the WE.

This study proposes a Bayesian approach to estimating the parameters of IWE. For such a complex model, the Markov chain Monte Carlo technique is conducted to save computational time. Lindley's approximation estimates the parameters for multiple complicated integrations under the Bayesian framework. On the other hand, Restogi and Merovci (2018) estimated three parameters of Weibull Rayleigh distribution with this method. Similar work had been conducted by Nazri *et al*. (2021).

Materials and Methods

Inverted Weighted Exponential Distribution Function

The two parameters of the IWE Probability Density Function (pdf) is given as:

$$g(x,\theta) = \frac{(\alpha+1)}{\alpha}\beta e^{-\beta x}\left(1 - e^{-\alpha\beta x}\right), \quad \text{where } \theta = (\alpha, \beta).$$

The Cumulative Density Function (CDF) is obtained by integrating the pdf. The derivation of CDF of IWE distribution is as follows:

$$G(x,\theta) = 1 - \frac{1}{\alpha} e^{-\beta x}\left(\alpha + 1 - e^{-\alpha\beta x}\right).$$

The likelihood function of IWE is:

$$L(x,\theta) = \prod_{i=1}^{n} \left(\frac{(\alpha+1)}{\alpha} \beta e^{-\beta x_i}\left(1 - e^{-\alpha\beta x_i}\right)\right).$$

The log-likelihood of IWE is:

$$\ell(x,\theta) = n \ln\left(\frac{\alpha+1}{\alpha}\right) + n\ln(\beta) - \beta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left(1 - e^{-\alpha\beta x_i}\right).$$

Bayesian Estimation

Bayes estimators cannot be obtained in closed form. Lindley's approximation is used to calculate the mean of Bayesian estimation. The loss function is a function of a difference between estimated and true data values. According to Dey (2012), loss function estimation and prediction might be problematic since no specific analytical procedure exists to identify the appropriate loss function. In this situation, the Squared Error Loss Function (SELF), which is symmetric in nature, will be considered. According to Guure and Ibrahim (2014), the Bayes estimator u_{BS} of a function $u = (\alpha,\beta)$ of the unknown parameters α and β under the SELF-Bayes estimator is the posterior mean. Therefore, the SELF-Bayes estimator is:

$$\widehat{\theta_{BS}} = u + \frac{1}{2}[(u_{11}\sigma_{11} + u_{22}\sigma_{22})] + u_1 p_1 \sigma_{11} + u_2 p_2 \sigma_{22} + \frac{1}{2}[(L_{30}u_1\sigma_{11}^2) + (L_{03}u_2\sigma_{22}^2)],$$

where,

$$p_1 = -\frac{1}{\alpha}, p_2 = -\frac{1}{\beta},$$

$$\sigma_{11} = -L_{20}^{-1}, \sigma_{22} = -L_{02}^{-1},$$

where $u = \sigma^2$,

$$u_1 = \frac{\delta u}{\delta\alpha} = 1, u_{11} = \frac{\delta^2 u}{(\delta\alpha)^2} = 0,$$

$$u_2 = \frac{\delta u}{\delta\beta} = 0, u_{22} = \frac{\delta^2 u}{(\delta\beta)^2} = 0,$$

where $u = p$,

$$u_1 = \frac{\delta u}{\delta\alpha} = 0, u_{11} = \frac{\delta^2 u}{(\delta\alpha)^2} = 0,$$

$$u_2 = \frac{\delta u}{\delta\beta} = 1, u_{22} = \frac{\delta^2 u}{(\delta\beta)^2} = 0.$$

Since SELF is symmetric, an asymmetric loss function, which is a Linear Exponential (LINEX) loss function, is proposed for use. The LINEX loss function was introduced by Varian (1975). By minimising the posterior expectation of the LINEX loss function, the value of the posterior mean is:

$$\widehat{\theta_{BL}} = -\frac{1}{a}\ln E_\theta(e^{-a\theta}).$$

The LINEX loss function is obtained using the same Lindley procedure.

$$\widehat{u_{BL}} = u + \frac{1}{2}[(u_{11}\sigma_{11} + u_{22}\sigma_{22})] + u_1p_1\sigma_{11} + u_2p_2\sigma_{22} + \frac{1}{2}[(L_{30}u_1\sigma_{11}^2) + (L_{03}$$

when $u = e^{-a\alpha}$,

$$u_1 = \frac{\delta u}{\delta \alpha} = -ae^{-a\alpha}, u_{11} = \frac{\delta^2 u}{(\delta\alpha)^2} = a^2e^{-a\alpha},$$

$$u_2 = \frac{\delta u}{\delta \beta} = 0, u_{22} = \frac{\delta^2 u}{(\delta\beta)^2} = 0,$$

when $u = e^{-a\beta}$,

$$u_1 = \frac{\delta u}{\delta \alpha} = 0, u_{11} = \frac{\delta^2 u}{(\delta\alpha)^2} = 0,$$

$$u_2 = \frac{\delta u}{\delta \beta} = -ae^{-a\beta}, u_{22} = \frac{\delta^2 u}{(\delta\beta)^2} = a^2e^{-a\beta}.$$

Prior Distribution

Guure *et al*. (2012) stated that in Bayesian analysis, non-informative prior can be employed if the prior knowledge about the parameter is unavailable. Since there is not much prior knowledge of the parameters, the extension of Jeffreys' prior is used as follows:

$$u(\alpha,\beta) \propto \left(\frac{1}{\alpha\beta}\right).$$

Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) is a technique that defines values for the parameters of a model of a given distribution. MLE for α parameter is:

$$\ell'(x,\alpha) = \frac{n}{\alpha+1} - \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{\beta x_i e^{-\alpha\beta x_i}}{(1 - e^{-\alpha\beta x_i})}.$$

MLE for β parameter is:

$$\ell'(x,\beta) = \frac{n}{\beta} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\alpha x_i e^{-\alpha\beta x_i}}{(1 - e^{-\alpha\beta x_i})}.$$

Simulation Study

A detailed simulation study compares the performance of Bayesian estimators and non-Bayesian estimators, i.e., MLE. The data is generated using the inverse transform of the CDF of the IWE distribution. The different values of parameters of IWE distribution considered α = 0.5, 0.6, 2.0, 3.0 and β = 0.3, 0.7, 2.0, 3.0. In addition, a sample size of n = 25, 50, 100 is chosen to represent small, medium, and large data sets (Ashour *et al*., 2022). These were iterated 5,000 times. A comparison of the SELF, LINEX loss functions, and MLE is made using Root Mean Square Error (RMSE). Conclusions are provided regarding the behaviour of the estimators.

Results and Discussions

The values of the constant under the LINEX loss function are chosen to be $a = (0.6, -0.6)$. We compare the Bayesian and non-Bayesian estimations by monitoring their RMSE values. Table 1 summarises the value of RMSE corresponding to estimation using the SELF, LINEX loss function, and MLE. In general, a higher sample size, n will lead to more precise estimation, which is evidenced in smaller RMSE values. This is due to the large sample properties of the MLE and Bayes estimation. Estimations using LINEX have the biggest RMSE values, implying that they are not as good as estimations from SELF and MLE. Figures 1 and 2 plot and compare the RMSE values for the shape and scale parameters, α and β respectively. These plots are for four sets of different α and β values and different sample sizes, n. When estimating α, MLE produces lower RMSE regardless of sample sizes, n, except when $\alpha = 0.5$ and $\beta = 2.0$ (Figure 1). As expected, we observe a declining trend in the RMSE values and n increases. However, the best estimation of the scale parameter, β is from using the Bayesian approach with SELF (Figure 2).

Table 1: Root mean square error (RMSE) of parameter estimates $\hat{a}$ and $\hat{\beta}$, with respect to SELF, LINEX loss function, and MLE at sample size, n = 25, 50, 100

n	α	β	**SELF**		**LINEX [0.6]**		**LINEX [-0.6]**		**MLE**	
			$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
25	0.5	2.0	0.00258	0.00179	0.00605	0.04468	0.01192	0.11275	0.00467	0.01305
	0.6	0.3	0.00457	0.00044	0.00987	0.00059	0.01891	0.00232	0.00236	0.00067
	2.0	3.0	0.03840	0.00000	0.12972	0.10464	0.35141	0.23686	0.00046	0.00029
	3.0	0.7	0.09571	0.00029	0.38808	0.00567	0.95653	0.01523	0.08989	0.00527
50	0.5	2.0	0.00111	0.00046	0.00282	0.02315	0.00569	0.05991	0.00235	0.00641
	0.6	0.3	0.00184	0.00012	0.00442	0.00041	0.00876	0.00130	0.00115	0.00034
	2.0	3.0	0.01920	0.00000	0.06360	0.05314	0.15133	0.12959	0.00019	0.00010
	3.0	0.7	0.04582	0.00008	0.17946	0.00289	0.57886	0.00773	0.04428	0.00259
100	0.5	2.0	0.00051	0.00012	0.00136	0.01178	0.00278	0.03094	0.00112	0.00301
	0.6	0.3	0.00081	0.00003	0.00209	0.00024	0.00422	0.00068	0.00057	0.00017
	2.0	3.0	0.00960	0.00000	0.03150	0.02678	0.07109	0.06815	0.00020	0.00010
	3.0	0.7	0.02245	0.00002	0.08659	0.00146	0.22307	0.00389	0.02164	0.00126

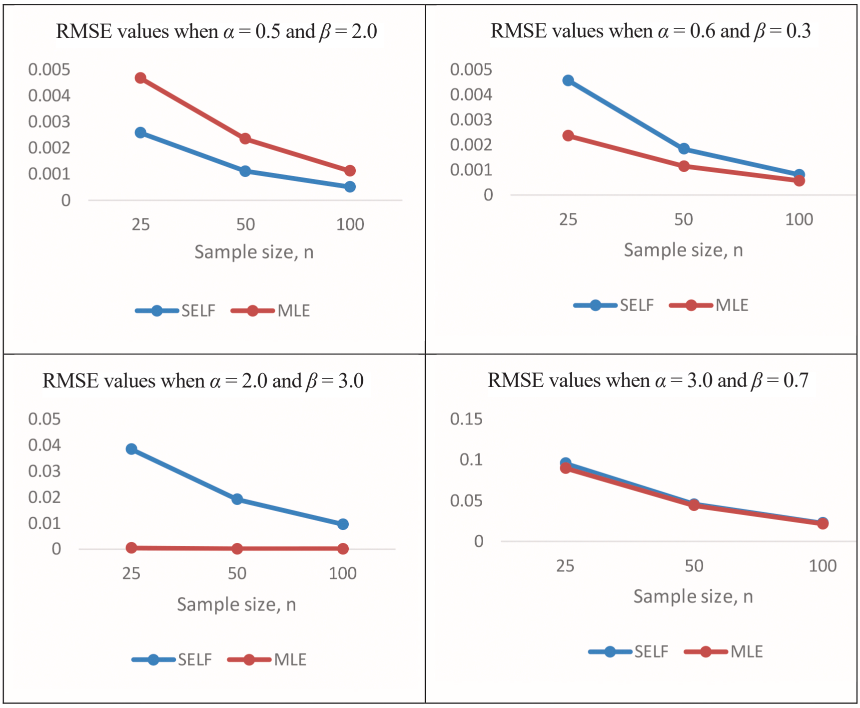

Figure 1: Line plots for root mean square error (RMSE) of parameter estimates $\hat{\alpha}$ with respect to the SELF-loss function and MLE at sample size, $n = 25, 50, 100$

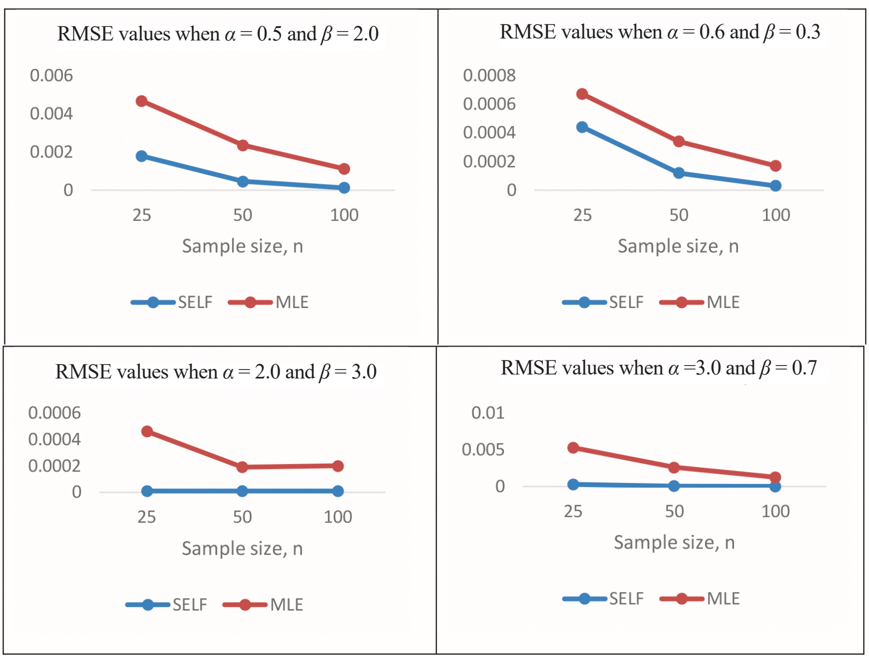

Figure 2: Line plots for Root mean square error (RMSE) of parameter estimates $\hat{\beta}$ with respect to the SELF-loss function and MLE at sample size, $n = 25, 50, 100$

Conclusions

In conclusion, MLE provides a better estimate for the shape parameter of the IWE distribution. Nevertheless, as observed from the results, the scale parameter benefitted from using the Bayesian approach with SELF. The estimators also perform better with a larger sample size, n.

Acknowledgements

This research is supported by the Ministry of Education (MOE) through the Fundamental Research Grant Scheme (FRGS/1/2019/STG06/UPM/02/10).

Conflict of Interest Statement

The authors declare that they have no conflict of interest.

References

Ashour, A. R., Ibrahim, N. A., Khaleel, M. A., & Oguntunde, P. E. (2022). Parameter estimation for Weibull Burr type X model with right censored data. *Mathematics and Statistics*, *10*(6), 1166-1177. https://doi.org/10.13189/ms.2022.100603

Bhaumik, D. K., Kapur, K., & Gibbons, R. D. (2009). Testing parameters of a gamma distribution for small samples. *Technometrics*, *51*(3), 326-334. https://doi.org/10.1198/tech.2009.07038

Dey, S. (2012). Bayesian estimation of the parameter and reliability function of an inverse Rayleigh distribution. *Malaysian Journal of Mathematical Sciences*, *6*(1), 113-124.

Gupta, R. D., & Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, *43*(6), 621-634. https://doi.org/10.1080/02331880802605346

Guure, C. B., Ibrahim, N. A., & Ahmed, A. O. M. (2012). Bayesian estimation of two-parameter weibull distribution using extension of Jeffreys' prior information with three loss functions. *Mathematical Problems in Engineering*, *2012*, Article 589640.

Guure, C. B., & Ibrahim, N. A. (2014). Approximate Bayesian estimates of Weibull parameters with Lindley's method. *Sains Malaysiana*, *43*(9), 1433-1437.

Nazri, F. S., Zulkafli, H. S., & Abd Rahman, N. H. (2021). Bayesian estimation of two-parameters rayleigh-logarithmic using lindley approximation. *Menemui Matematik (Discovering Mathematics)*, *43*(1), 61-67.

Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O., & Odetunmibi, O. A. (2017). A new generalisation of the lomax distribution with increasing, decreasing, and constant failure rate. *Modelling and Simulation in Engineering*, *2017*, 1-6. https://doi.org/10.1155/2017/6043169

Oguntunde, P. E., Ilori, K. A., & Okagbue, H. I. (2018). The inverted weighted exponential distribution with applications. *International Journal of Advance Applied Sciences*, *5*, 46-50.

Rastogi, M. K., & Merovci, F. (2017). Bayesian estimation for parameters and reliability characteristic of the Weibull Rayleigh distribution. *Journal of King Saud University - Science*, *30*(4), 472-478. https://doi.org/10.1016/j.jksus.2017.05.008

Varian, H. R. (1975). A Bayesian approach to real estate assessment (pp. 195-208). In (S. E. Fienberg, & A. Zellner (Eds), *Studies in Bayesian Econometrics and Statistics: In Honor of Leonard J. Savage*. New York.