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ON THE LOWER SPECTRUM OF THE TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATOR

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ABSTRACT

In this paper, it is established that if the lower edge of the spectrum is an isolated point of the Schrödinger operator of a system of two particles on a lattice, then it is a simple eigenvalue and a strictly positive eigenvector corresponds to it, under certain conditions with respect to the dispersion functions and potential.

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Introduction

Discrete Schrödinger operators corresponding to the Hamiltonians of systems of one and two quantum particles on an integer lattice are studied in [1, 2, 3, 6-10]. The earliest results relating positivity and non-degeneracy of an eigenvalue are found in the fundamental theorem of Perron and Frobenius: A finite matrix with strictly positive elements has its spectral radius as an eigenvalue of unit multiplicity, and the corresponding eigenvector is strictly positive. Note that in the Perron-Frobenius theorem, the matrix does not have to be self-adjoint. This theorem first appeared in [2] and then in [3]. The generalisation of the Perron-Frobenius theorem to quantum systems is obtained in the work of Glimm-Jaffe [4]. An application to non-relativistic systems was defined in the work of Simon and Hoegh Krohn [5].

In this scientific work, it is shown that the resolvent of the unperturbed energy operator of a two-particle system on a d-dimensional lattice is positivity-enhancing for all values of the total quasi-momentum of the system $k \in (-\pi;\pi)^d, z < \mathrm{e}_m(k)$ and positivity-preserving for $k \in T^d \setminus (-\pi;\pi)^d, z < \mathrm{e}_m(k)$.

In addition, it has been established that if the lower edge is an isolated point of the spectrum of the Schrödinger operator of a system of two particles on a d – d-dimensional lattice, then for $k \in (-\pi;\pi)^d$ it is a simple eigenvalue and a strictly positive eigenvector corresponds to it, and for $k \in T^d \setminus (-\pi;\pi)^d$, it is a multiple eigenvalue and a positive eigenvectors corresponds to it.

Energy Operator of Two Particles in Coordinate and Momentum Representations

Let be the set of integers. We denote the d-dimensional lattice by $\mathbb{Z}^d$, and $\ell_2(\mathbb{Z}^d)$ is the Hilbert space of quadratically summable functions defined on $\mathbb{Z}^d$. Let $\mathbb{T}^d = (R/2\pi\mathbb{Z})^d = (-\pi;\pi]^d$ be a d-dimensional torus. In this torus, the operations of addition and multiplication by the real number of elements of the set $(-\pi;\pi]^d \subset \mathbb{R}^d$ are understood everywhere as operations on $\mathbb{R}^d$ modulo $(2\pi\mathbb{Z})^d$ also $L_2(\mathbb{T}^d)$ is the Hilbert space of quadratically integrable functions defined on $\mathbb{T}^d$.

The unperturbed energy operator $\grave{H}_0^{(2)}$ of a system of two quantum particles on a d-dimensional lattice $\mathbb{Z}$ is usually associated with the following operator in the Hilbert space $\ell_2((\mathbb{Z}^d)^2)$:

$$\grave{H}_0^{(2)} = -\frac{1}{2m_1}\grave{\Delta}_1 - \frac{1}{2m_2}\grave{\Delta}_2,$$

where $m_1, m_2 > 0$ are the masses of the particles and

$$\grave{\Delta}_1 = \grave{\Delta} \otimes I,\ \grave{\Delta}_2 = I \otimes \grave{\Delta}.$$

Here, I is the identity operator in $\ell_2(\mathbb{Z}^d)$, and $\grave{\Delta}$ is the standard lattice Laplacian:

$$\grave{\Delta} = \sum_{j=1}^{d} \left(\grave{T}(e_j) + \grave{T}(-e_j) - \grave{T}(0)\right).$$

Here, $e_j, j = 1, \dots, d$ are unit vectors along the j-th direction of $\mathbb{Z}^d$, also $\grave{T}(s)$ is the shift operator by s, defined by the following formula:

$$\left(\grave{T}\grave{f}\right)(s) = \grave{f}(x+s),\ \grave{f} \in \ell_2(\mathbb{Z}^d),\ \ s \in \mathbb{Z}^d. \tag{1}$$

More precisely,

$$\left(\grave{\Delta}\grave{f}\right)(x) = \sum_{j=1}^{d} \left(\grave{f}(x+e_j) + \grave{f}(x-e_j) - 2\grave{f}(x)\right),\ \grave{f} \in \ell_2(\mathbb{Z}^d).$$

Note that the operator $\grave{H}_0^{(2)}$ is a bounded self-adjoint operator.

It is easy to verify that for each $s \in \mathbb{Z}^d$ the operator $\grave{T}(s)$, defined by formula (1) is unitarily equivalent to the operator $T(s)$ in the space $L_2(\mathbb{T}^d)$, i.e.:

$$(T(s)f)(p) = e^{-i(s,p)}f(p).$$

The energy operator $H_0^{(2)}$ (in the coordinate representation) of a system of two quantum particles moving on a d-dimensional lattice $\mathbb{Z}^d$ with pair interaction $\grave{V}_{12}$ is a bounded self-adjoint operator in the Hilbert space $\ell_2((\mathbb{Z}^d)^2)$ of the form

$$\grave{H}^{(2)} = \grave{H}_0^{(2)} - \grave{V}_{12}.$$

Here

$$\left(\grave{V}_{12}\grave{f}\right)(x,y) = \grave{v}(x-y)\grave{f}(x,y),\ \grave{f} \in \ell_2((\mathbb{Z}^d)^2),$$

where $\grave{v}: \mathbb{Z}^d \to \mathbb{R}$ is a bounded function.

In addition, assume that the function $\grave{v}$ satisfies the following assumption:

Assumption 1. Assume that $\grave{v}(\cdot) \neq 0$ is non-negative and $\grave{v} \in \ell_1(\mathbb{Z}^d)$.

Lemma 1. Let Assumption 1 be satisfied. Then, the operator $\grave{V}_{12}$ is bounded, self-adjoint and positive, and $\grave{H}^{(2)}$ is a bounded self-adjoint operator.

The transition to the impulse representation is carried out using the Fourier transform. The standard Fourier transform $F: L_2((\mathbb{T}^d)^2) \to \ell_2((\mathbb{Z}^d)^2)$ is determined by the following formula:

$$(Ff)(x,y) = (2\pi)^{-d} \int_{(\mathbb{T}^d)^2} f(t,s) e^{-i(x,t)-i(y,s)} dtds$$

and inverse mapping $F^{-1}: \ell_2((\mathbb{Z}^d)^2) \to L_2((\mathbb{T}^d)^2)$,

$$\left(F^{-1}\hat{f}\right)(p,q) = (2\pi)^{-d} \sum_{x,y\in\mathbb{Z}^d} \hat{f}(x,y) e^{i(x,p)+i(y,q)}.$$

Note that the values of the function $\hat{f}(x,y) = (Ff)(x,y)$ are called the Fourier coefficients of the function $f \in L_2((\mathbb{T}^d)^2)$ in the orthonormal basis $\{\psi_x(p)\psi_y(q)\}_{x,y\in\mathbb{Z}^d}$. In addition, $F = \mathcal{F} \otimes \mathcal{F}$. In this case, the momentum representation of the energy operator of the two-particle system acts in the Hilbert space $L_2((\mathbb{T}^d)^2)$ and has the form:

$$H^{(2)}: L_2((\mathbb{T}^d)^2) \to L_2((\mathbb{T}^d)^2),\ H^{(2)} = H_0^{(2)} - V_{12},$$

where

$$H^{(2)} = F^{-1}\grave{H}^{(2)}F,\quad H_0^{(2)} = F^{-1}\grave{H}_0^{(2)}F,\quad V_{12} = F^{-1}\grave{V}_{12}F,$$

$$\left(H_0^{(2)}f\right)(p,q) = E(p,q)f(p,q),\ (V_{12}f)(p,q) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} v(s-q) f(p+q-s,s) ds.$$

Here

$$E(p,q) = \frac{1}{m_1}\varepsilon(p) + \frac{1}{m_2}\varepsilon(q),\ \varepsilon(p) = \sum_{j=1}^{d} \left(1 - \cos\, p_j\right),$$

$$v(p) = (2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} \grave{v}(n) e^{i(n,p)},\ p = (p_1, \dots, p_d) \in \mathbb{T}^d.$$

Lemma 2. The spectrum of the operator $\grave{H}_0^{(2)}$ is absolutely continuous and the following equalities hold [7]:

$$\sigma\left(\grave{H}_0^{(2)}\right) = \sigma_{ac}\left(\grave{H}_0^{(2)}\right) = \sigma\left(H_0^{(2)}\right) = [0; 2d(m_1^{-1} + m_2^{-1})].$$

Direct Integral Expansion of the Energy Operator

In the momentum representation, the family $\{\grave{T}(n), n \in \mathbb{Z}^d\}$ (1) is unitarily equivalent to the family $\{T(n), n \in \mathbb{Z}^d\}$, where

$$(T(n)f)(p,q) = e^{-i(n,p+q)} f(p,q),\ f \in L_2((\mathbb{T}^d)^2).$$

The commutation relation $\grave{H}^{(2)}\grave{T}(n) = \grave{T}(n)\grave{H}^{(2)}$ generates the commutation (relation) $H^{(2)}T(n) = T(n)H^{(2)}, n \in \mathbb{Z}^d$. Consequently, the Abelian group of unitary operators $\{T(n), n \in \mathbb{Z}^d\}$ commutes with the operator $H^{(2)} = H_0^{(2)} - V_{12}$. Therefore, the space $L_2((\mathbb{T}^d)^2)$ and the operator $H^{(2)}$ are expanded into direct Neumann integrals:

$$L_2((\mathbb{T}^d)^2)=\int_{\mathbb{T}^d}\oplus L_2(\mathbb{F}_k)dk,\quad H^{(2)}=\int_{\mathbb{T}^d}\oplus \grave{h}(k)dk,$$

where

$$\mathbb{F}_k=\{(p,q)\in(\mathbb{T}^d)^2, p+q=k\},\quad k\in\mathbb{T}^d.$$

Here

$$\grave{h}(k): L_2(\mathbb{F}_k)\to L_2(\mathbb{F}_k),\quad \grave{h}(k)=\grave{h}_0(k)-\grave{V}.$$

Now we give the actions of the operators $\grave{h}_0(k)$ and $\grave{V}$ on the element $f\in L_2(\mathbb{F}_k)$:

$$\left(\grave{h}_0 kf\right)(p,k-p)=\grave{E}_k(p)f(p,k-p),\ \left(\grave{V}f\right)(p,k-p)=(2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d}v(p-s)f(s,k-s)ds,$$

where

$$\grave{E}_k(p)=\frac{1}{m_1}\varepsilon(p)+\frac{1}{m_2}\varepsilon(k-p).$$

We will find another analytical representation for the function $\grave{E}_k(p)$, which we will need later:

$$\begin{aligned}\grave{E}_k(p)&=\sum_{j=1}^{d}\left[\frac{1}{m_1}\left(1-\cos p_j\right)+\frac{1}{m_2}\left(1-\cos\left(p_j-k_j\right)\right)\right]\\&=\sum_{j=1}^{d}\left[\mu(0)-\left|\mu\left(k_j\right)\right|\cos\left(p_j-\varphi\left(k_j\right)\right)\right],\end{aligned}$$

where

$$\mu\left(k_j\right)=\frac{1}{m_1}+\frac{1}{m_2}e^{-ik_j},\ \varphi\left(k_j\right)=\arg\mu\left(k_j\right), k_j\in\mathbb{T}. \tag{2}$$

The function $\grave{E}_k(p)$ can be represented in the following form:

$$\grave{E}_k(p)=d\mu(0)-\sum_{j=1}^{d}\left|\mu\left(k_j\right)\right|\cos\left(p_j-\varphi\left(k_j\right)\right), \tag{3}$$

where the coefficients $\left|\mu\left(k_j\right)\right|$ are defined by the following equalities:

$$\left|\mu\left(k_j\right)\right|=\left|\frac{1}{m_1}+\frac{1}{m_2}e^{-ik_j}\right|=\sqrt{\frac{1}{m_1^2}+\frac{1}{m_2^2}+\frac{2\cos\ k_j}{m_1 m_2}}.$$

Equation (3) gives the following representation for $\grave{E}_k(\cdot)$:

$$\grave{E}_k\left(p+\varphi(k)\right)=E_k(p)=d\mu(0)-\sum_{j=1}^{a}\left|\mu\left(k_j\right)\right|\cos p_j. \tag{4}$$

Now, we introduce the following unitary mapping:

$$U_k: L_2(\mathbb{F}_k) \to L_2(\mathbb{T}^d),$$

which is determined by the following formula

$$(U_k f)(p) = f(p, k - p).$$

In this case, the operator

$$h(k) = U_k \grave{h}(k) U_k^{-1}$$

is unitarily equivalent to the operator $\grave{h}(k)$, where

$$U_k: L_2(\mathbb{F}_k) \to L_2(\mathbb{T}^d), \qquad (U_k^{-1} f)(p, q) = f\left(\frac{p - q + k}{2}\right).$$

We will mainly study the operator $h(k)$ and its coordinate representation. The operator $h(k)$ has the form:

$$h(k) = h_0(k) - V.$$

The actions of the operators $h_0(k)$ and V on the element $f \in L_2(\mathbb{T}^d)$ have the following form:

$$(h_0(k)f)(p) = E_k(p) f(p), \ (Vf)(p) = \int_{\mathbb{T}^d} v(p - t) f(t) dt,$$

where

$$v(p) = (2\pi)^{-\frac{d}{2}} \sum_{s \in \mathbb{Z}^d} \grave{v}(s) e^{i(s,p)}.$$

It is known that the spectrum of the operator V coincides with the set $\{0, \grave{v}(n), n \in \mathbb{Z}^d\}$, and the numbers $\grave{v}(n)$ are the eigenvalues of the operator V. Since $\grave{v} \in \ell_1(\mathbb{Z}^d)$, then V is an operator with a trace (nuclear), in particular, a compact operator. Since the perturbation $V = h(k) - h_0(k)$ is compact, then according to the Weil theorem, we obtain that $\sigma_{\text{ess}}\left(h(k)\right) = \sigma_{ess}\left(h_0(k)\right)$. For convenience, we take the following designations:

$$\mathrm{e}_m(k) = \min_{q \in \mathbb{T}^d} E_k(q), \ \mathrm{e}_M(k) = \max_{q \in \mathbb{T}^d} E_k(q)$$

From (4) it follows

$$\mathrm{e}_m(k) = d\mu(0) - \sum_{j=1}^{d} \left|\mu(k_j)\right|, \quad \mathrm{e}_M(k) = d\mu(0) + \sum_{j=1}^{d} \left|\mu(k_j)\right|.$$

Lemma 3. The spectrum of the operator $h_0(k)$ coincides with the segment $[\mathrm{e}_m(k); \mathrm{e}_M(k)]$. In addition [7],

$$\sigma_{\text{ess}}\left(h_0(k)\right) = \sigma\left(h_0(k)\right) = [\mathrm{e}_m(k); \mathrm{e}_M(k)].$$

Two-Particle Discrete Schrödinger Operator

Let us denote by $\grave{h}(k), k \in \mathbb{T}^d$ the coordinate representation of the operator $h(k), k \in \mathbb{T}^d$, i.e.

$$\grave{h}(k) = \mathcal{F}h(k)\mathcal{F}^{-1}. \tag{5}$$

The standard Fourier transform $\mathcal{F}: L_2(\mathbb{T}^d) \to \ell_2(\mathbb{Z}^d)$, which is defined by the formula

$$(\mathcal{F}f)(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{T}^d} e^{-i(x,t)} f(t)dt$$

and the inverse Fourier transform $\mathcal{F}^{-1}: \ell_2(\mathbb{Z}^d) \to L_2(\mathbb{T}^d)$:

$$\left(\mathcal{F}^{-1}\grave{f}\right)(p) = (2\pi)^{-\frac{d}{2}} \sum_{s\in\mathbb{Z}^d} e^{i(s,p)} \grave{f}(s).$$

In the physical literature, the parameter $k \in \mathbb{T}^d$ is called the total quasi-momentum of a system of two particles and the corresponding operators $\grave{h}(k), k \in \mathbb{T}^d$ are called layers of the operator $\grave{H}^{(2)}$. The layer operators $\grave{h}(k), k \in \mathbb{T}^d$ in equality (5) are defined by the following formula:

$$\grave{h}(k) = \grave{h}_0(k) - \grave{V}.$$

Now, we find analytical representations for the unperturbed operator $\grave{h}_0(k)$ and the perturbation operator $\grave{V}$. The following statement holds.

Lemma 4. The action of the operator $\grave{h}_0(k)$ on the element $f \in \ell_2(\mathbb{Z}^d)$ has the form

$$\left(\grave{h}_0(k)\grave{f}\right)(x) = \frac{1}{2}\sum_{j=1}^{d} \left(2\mu(0)I\grave{f}(x) - |\mu(k_j)| \left(\grave{f}(x+e_j) - \grave{f}(x-e_j)\right)\right),$$

where $\mu(\cdot)$ is determined by formula (2).

Lemma 5. The action of the operator $\grave{V}$ on the element $f \in \ell_2(\mathbb{Z}^d)$ has the form

$$\left(\grave{V}\grave{f}\right)(x) = \grave{v}(x)\grave{f}(x).$$

Under Assumption 1, the perturbation $\grave{V}$ of the operator $\grave{h}_0(k), k \in \mathbb{T}^d$ is compact and, therefore, according to Weil theorem on the essential spectrum, the essential spectrum of the operator $h(k)$ consists of the following segment on the real axis:

$$\sigma_{\text{ess}}\left(h(k)\right) = [\mathrm{e}_m(k); \mathrm{e}_M(k)].$$

Now, we find the explicit form of the resolvent of the operator $\grave{h}_0(k)$. The resolvent of the operator $\grave{h}_0(k)$

$$\grave{R}_0(k,z) = \left(\grave{h}_0(k) - zI\right)^{-1}, \ z \in \mathbb{C} \setminus [\mathrm{e}_m(k); \mathrm{e}_M(k)]$$

is determined by the following formula (formula 2.7 in [9]):

$$\left(\check{R}_0(k,z)\check{f}(x)\right) = \sum_{s\in\mathbb{Z}^d} \check{r}_0(x-s,k;z)\check{f}(s),\ \check{f} \in \ell_2(\mathbb{Z}^d),$$

where

$$\check{r}_0(x,k;z) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \frac{e^{-i(x,t)}}{E_k(t)-z}dt, x\in\mathbb{Z}^d, z\in\mathbb{C}\setminus[\mathfrak{e}_m(k);\mathfrak{e}_M(k)].$$

We will assume that the masses of the particles are equal, i.e. $m_1 = m_2 = m$. The following lemma plays an important role in proving the main results of this section.

Lemma 6. Let $z < \mathfrak{e}_m(k)$. Then the following statements hold:

(a) if $k \in (-\pi;\pi)^d$, then the function $\check{r}_0(\cdot,k;z)$ is strictly positive in $\mathbb{Z}^d$;

(b) if $k \in \mathbb{T}^d \setminus (-\pi;\pi)^d$, then the function $\check{r}_0(\cdot,k;z)$ is positive in $\mathbb{Z}^d$.

Proof. (a) We consider the following function:

$$r_0(p,k;z) := \frac{1}{E_k(p)-z} = \frac{1}{\frac{2}{m}\sum_{j=1}^{d}\left(1-\cos p_j\cos\frac{k_j}{2}\right)-z}, p=(p_1,\dots,p_d)\in\mathbb{T}^d, z<\mathfrak{e}_m(k),$$

for $k\in(-\pi;\pi)^d$. Note that for each $k_j\in(-\pi;\pi)$ we have $\cos\frac{k_j}{2}>0$. Because, $\frac{2d}{m}-z > \frac{2}{m}\sum_{j=1}^{d}\cos p_j\cos\frac{k_j}{2}$, that $\left|\frac{\frac{2}{m}\sum_{j=1}^{d}\cos p_j\cos\frac{k_j}{2}}{\frac{2d}{m}-z}\right|<1$, therefore the function $r_0(\cdot,k;z)$ can be expanded into a series, i.e.:

$$r_0(p,k;z) = \sum_{n=0}^{\infty}\frac{1}{\left(\frac{2d}{m}-z\right)^{n+1}}\left(\frac{2}{m}\sum_{j=1}^{d}\cos p_j\cos\frac{k_j}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{\left(\frac{2}{m}\right)^n}{\left(\frac{2d}{m}-z\right)^{n+1}}\sum_{0\le l_1,\dots,l_d\le n} C_n^{l_1,\dots,l_d}\prod_{j=1}^{d}\left(\cos p_j\cos\frac{k_j}{2}\right)^{l_j}, \qquad (6)$$

where

$$C_n^{l_1,\dots,l_d} = \frac{n!}{l_1!\dots l_d!}.$$

Using Euler's formula, we obtain

$$(\cos p)^k = \left(\frac{e^{ip}+e^{-ip}}{2}\right)^k = \frac{1}{2^k}\sum_{m=0}^{k} C_k^m e^{ip(2m-k)}. \qquad (7)$$

We will prove that all Fourier coefficients $\grave{r}_0(x,k;\cdot)$ of the function $\grave{r}_0(p,k,\cdot)$ are non-negative. Substituting (7) into (6), we obtain the following representation for $r_0(p;k,\cdot)$:

$$r_0(p;k,z)=\sum_{n=0}^{\infty}\frac{\left(\frac{2}{m}\right)^n}{\left(\frac{2d}{m}-z\right)^{n+1}}\sum_{\substack{0\le l_1,\dots,l_d\le n,\\ l_1+\dots+l_d=n}}C_n^{l_1,\dots,l_d}\prod_{j=1}^{d}\left(\frac{1}{2}\cos\frac{k_j}{2}\right)^{l_j}\sum_{m_j=0}^{l_j}C_{l_j}^{m_j}e^{ip_j(2m_j-l_j)}. \quad (8)$$

Using formula (8), we write the coefficients $\grave{r}_0(x,k;z)$ in the form:

$$\grave{r}_0(x,k;z)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{e^{-i(x,t)}}{E_k(t)-z}dt$$

$$=\frac{1}{(2\pi)^d}\sum_{n=0}^{\infty}\frac{\left(\frac{2}{m}\right)^n}{\left(\frac{2d}{m}-z\right)^{n+1}}\sum_{0\le l_1,\dots,l_d\le n}C_n^{l_1,\dots,l_d}\prod_{j=1}^{d}\left(\frac{1}{2}\cos\frac{k_j}{2}\right)^{l_j}\sum_{m_j=0}^{l_j}C_{l_j}^{m_j}\int_{\mathbb{T}^d}e^{it_j(2m_j-l_j-x_j)}dt_j.$$

According to the positivity of $\frac{2d}{m}-z, \cos\frac{k_j}{2}$, for $k_j\in(-\pi;\pi)$ and using the following formula

$$(2\pi)^{-d}\int_{\mathbb{T}^d}e^{i(m,t)}dt=\begin{cases}1, & m=0\\ 0, & m\neq 0,\ m\in\mathbb{Z}^d\end{cases}, \quad (9)$$

we have $\grave{r}_0(x,k;z)\ge 0$.

Now, we will prove that for any $x\in\mathbb{Z}^d$, the function $\grave{r}_0(x,k;z)$ is strictly positive. For each $x\in\mathbb{Z}^d$ there exist numbers $l_j^0\in\{0,1,2,\dots\}, m_j^0\in\{0,1,2,\dots,l_j^0\}$ such that $2m_j^0-l_j^0-x_j=0$. Then we have

$$\grave{r}_0(x,k;z)\ge\frac{\left(\frac{2}{m}\right)^{l_j^0}}{\left(\frac{2d}{m}-z\right)^{l_j^0+1}}C_{l_j^0}^{m_j^0}>0.$$

(b) First, let us assume that $k=(\pi,\dots,\pi)\in\mathbb{T}^d$. Then, we have

$$\grave{r}_0(x,\pi;z)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{e^{-i(x,t)}}{E_\pi(t)-z}dt=\frac{1}{(2\pi)^d}\frac{1}{\frac{2d}{m}-z}\int_{\mathbb{T}^d}e^{-i(x,t)}dt\ge 0.$$

Moreover, $\grave{r}_0(0,\pi;z)>0$ and $\grave{r}_0(x,\pi;z)\ge 0$, for $x\neq 0$.

Now, let us assume that l coordinates of the vector $k\in\mathbb{T}^d$ are equal to π, i.e.:

$$k=\grave{k}=(\pi,\dots,\pi,k_{l+1},\dots,k_d), k_j\in(-\pi;\pi),\ j=l+1,\dots,d.$$

In that case, we have

$$\grave{r}_0(x,\grave{k};z)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\frac{e^{-i(x,t)}}{E_{\grave{k}}(t)-z}dt=\frac{1}{(2\pi)^d}\frac{1}{\frac{2d}{m}-z}\int_{\mathbb{T}^l}e^{-i(x',t)}dt\sum_{n=0}^{\infty}\frac{1}{\left(\frac{2}{m}\right)^n\left(\frac{2d}{m}-z\right)^n}\times$$

$$\times\sum_{0\le s_{l+1},\dots,s_d\le n}C_n^{s_{l+1},\dots,s_d}\prod_{j=l+1}^{d}\left(\frac{1}{2}\cos\frac{k_j}{2}\right)^{s_j}\sum_{m_j=0}^{s_j}C_{s_j}^{m_j}\int_{\mathbb{T}^{d-l}}e^{it_j(2m_j-s_j-x_j)}dt_j\ge 0,$$

where $x'\in\mathbb{T}^l$. According to the formula (9), the function $\grave{r}_0(x,\grave{k};z)$ is positive. The lemma is proved.

Theorem 1. Let $z<\mathrm{e}_m(k)$. Then, the following statements hold:

(a) *if* $k\in(-\pi;\pi)^d$, then the operator $\grave{R}_0(k,z)$ is positivity-enhancing in $\ell_2(\mathbb{Z}^d)$;

(b) if $k\in\mathbb{T}^d\setminus(-\pi;\pi)^d$, then the operator $\grave{R}_0(k,z)$ is positivity-preserving in $\ell_2(\mathbb{Z}^d)$.

Proof. By the condition of the theorem, $z<\mathrm{e}_m(k)$. It is sufficient to prove that if $k\in(-\pi;\pi)^d$, then $\grave{R}_0(k,z)\grave{\psi}_y$ is strictly positive and if $k\in\mathbb{T}^d\setminus(-\pi;\pi)^d$, then $\grave{R}_0(k,z)\grave{\psi}_y$ is positive for elements of the following canonical basis:

$$\grave{\psi}_y(x)=\begin{cases}1, & x=y\\ 0, & x\neq y\end{cases}. \tag{10}$$

Then, according to (10), we have

$$\left(\grave{R}_0(k,z)\grave{\psi}_y\right)(x)=\sum_{s\in\mathbb{Z}^d}\grave{r}_0(x-s;k,z)\psi_y(s)=\grave{r}_0(x-y;k,z),\ z<\mathrm{e}_m(k).$$

By Lemma 6, if $k\in(-\pi;\pi)^d$, then the function $\grave{r}_0(\cdot;k,z)$ is strictly positive and if $k\in\mathbb{T}^d\setminus(-\pi;\pi)^d$, then the function $\grave{r}_0(\cdot;k,z)$ is positive. The theorem is proved.

Theorem 2. Let $z_0(k)=\inf\sigma\left(\grave{h}(k)\right)<\mathrm{e}_m(k)$ and $m_1=m_2$. Then, the following statements hold:

(a) if $k\in(-\pi;\pi)^d$, then $z_0(k)$ is a non-degenerate eigenvalue of $\grave{h}(k)$ and the corresponding eigenvector can be chosen strictly positive.

(b) *if* $k\in\mathbb{T}^d\setminus(-\pi;\pi)^d$, then for any $n\in\mathbb{N}$ there exists a potential $\grave{v}(x)$, such that $z_0(k)$ is an multiple eigenvalue of $\grave{h}(k)$ and the corresponding linearly independent eigenvectors can be chosen positive.

Proof. If the conditions of the theorem are valid $(m_1=m_2=m)$, then the function $E_k(p)$ can be represented in the following form:

$$E_k(p)=\frac{2}{m}\sum^{d}\left(1-\cos\frac{k_j}{2}\cos p_j\right).$$

In this case, the corresponding dispersion function $\grave{E}_k(s)$ of the operator $\grave{h}(k)$ has the form:

$$\grave{E}_k(s) = \begin{cases} \dfrac{2d}{m}, & s = 0 \\ -\dfrac{1}{m}\cos\dfrac{k_j}{2}, & |s| = 1 \\ 0, & |s| > 1 \end{cases}.$$

If condition (a) of Theorem 2 is valid, then the dispersion function $\grave{E}_k(s)$ satisfies the condition of Assumption 2.2 [9]. Then, the proof of part (a) of Theorem 2 follows from Theorem 4.1 [9]. Now, let us assume that condition (b) of the theorem is valid. First, we prove in the case $d = 1$. In this case, $k = \pi$, and the operator $\grave{h}(\pi)$ acts according to the formula:

$$\left(\grave{h}(\pi)\grave{f}\right)(x) = \left(\frac{2}{m} - \grave{v}(x)\right)\grave{f}(x).$$

Suppose that $\grave{v}(1) = \grave{v}(2) = \cdots = \grave{v}(n) > 0$ and $\grave{v}(x) = 0$, for $x \in \mathbb{Z} \setminus \{1,2,\ldots,n\}$. Then, the number $z_0 = \inf\sigma\left(\grave{h}(\pi)\right) = \frac{2}{m} - \grave{v}(1)$ is an n-multiple eigenvalue of the operator $\grave{h}(\pi)$. In this case, the elements of the canonical basis $\grave{\psi}_1(x), \ldots, \grave{\psi}_n(x)$ can be chosen as positive linearly independent eigenfunctions.

Let us provide proof of part (b) of Theorem 2 in the case $d = 2$. In this case, three cases are possible:

(b1) $k_1 = \pi, k_2 = \pi$, i.e. $k = (\pi;\pi)$;

(b2) $k_1 = \pi, k_2 \in (-\pi;\pi)$, i.e. $k = (\pi; k_2)$;

(b3) $k_1 \in (-\pi;\pi), k_2 = \pi$, i.e. $k = (k_1;\pi)$.

The proof of case (b1) is similar to the proof of the theorem in the case $d = 1$.

Proof of case (b2). Let $k_1 = \pi, k_2 \in (-\pi;\pi)$. Consider the operator $\grave{h}(\pi; k_2)$. We introduce the subspaces L_y – one-dimensional subspaces spanned by the vector $\grave{\psi}_y$, where $\grave{\psi}_y$ is an element of the canonical basis defined by formula (10). In this case, the space $\ell_2(\mathbb{Z})$ is expanded into a direct sum

$$\ell_2(\mathbb{Z}) = \bigoplus_{y\in\mathbb{Z}} L_y. \tag{11}$$

The following expansion follows from (11):

$$\ell_2(\mathbb{Z}^2) = \bigoplus_{y\in\mathbb{Z}} L_y \otimes \ell_2(\mathbb{Z}).$$

Lemma 6 [7]. The subspace $\mathrm{L_y} \otimes \ell_2(\mathbb{Z})$ for any $\mathrm{y} \in \mathbb{Z}$ is invariant with respect to the operator $\grave{h}(\pi; k_2)$.

The restriction of the operator $\grave{h}(\pi; k_2)$ to the subspace $L_y \otimes \ell_2(\mathbb{Z})$ has the form

$$\grave{h}^{(y)}(\pi; k_2) = I \otimes \grave{h}^{(y)}(k_2),$$

where I is the identity operator in L_y and the operator

$$\grave{h}^{(y)}(k_2) = \grave{h}_0(k_2) - \grave{V}_y,$$

is a two-particle discrete operator in $\ell_2(\mathbb{Z})$. More precisely

$$\left(\grave{h}_0(k_2)\grave{f}\right)(s) = \frac{4}{m}\grave{f}(s) - \frac{1}{m}\cos\frac{k_2}{2}\left[\grave{f}(s+1) + \grave{f}(s-1)\right],$$

$$\left(\grave{V}_y\grave{f}\right)(s) = \grave{v}_y(s)\grave{f}(s), \;\; \grave{v}_y(s) = \grave{v}(y,s).$$

The number $z_0 = \inf\sigma\left(\grave{h}_0(k)\right)$ is an n-multiple eigenvalue of the operator $\grave{h}_0(k)$. We choose the potential $\grave{v}$ as follows

$$\grave{v}(y,s) = \begin{cases} 1, & s = x_i, i = 1, \dots, n \\ 0, & s \neq x_i \end{cases}.$$

If $y \neq x_i, i = 1, \dots, n$, then $\grave{h}^{(y)}(\pi, k_2) = \grave{h}(\pi, k_2)$ and the operators $\grave{h}^{(x_i)}(\pi, k_2), \grave{h}^{(x_j)}(\pi, k_2),\; i,j = 1, \dots, n$ are unitarily equivalent.

It is easy to verify that the operator $\grave{h}^{(x_i)}(\pi, k_2)$ has a unique non-degenerate eigenvalue z_0, lying to the left of the essential spectrum of the operator $\grave{h}^{(x_i)}(\pi, k_2)$. This eigenvalue corresponds to a positive eigenfunction

$$\grave{\psi}_{x_i}(y)\grave{f}(s) \in L_{x_i} \otimes \ell_2(\mathbb{Z}), i = 1, \dots, n.$$

According to the unitary equivalence of the operators $\grave{h}^{(x_i)}(\pi, k_2), i = 1, \dots, n$., the number z_0 is an eigenvalue of these operators. Thus, the number z_0 is an n-multiple eigenvalue of the operator $\grave{h}(\pi, k_2)$.

The proof of case (b3) is the same as the proof of case (b2).

The proof of the theorem in case $d \geq 3$ is given similarly.

Conclusions

It is shown that the resolvent of the unperturbed energy operator of one-particle corresponding to the system on a d-dimensional lattice enhances positivity for all values of z −below the edge of the essential spectrum. It is proved that if the lower edge is an isolated point of the spectrum of the Schrödinger operator of a one-particle on a d-dimensional lattice, then it is a simple eigenvalue and it corresponds to a strictly positive eigenvector, that is an analogue of a theorem of Perron-Frobenius type was proved.

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Authors' Contributions

All authors contributed to the study conception and design. All authors read and approved the final manuscript. Conceptualisation: Kuljanov Utkir; Methodology: Kuljanov Utkir, Kuljanov Jovlon; Formal analysis and investigation: Kuljanov Utkir, Kuljanov Jovlon; Writing - original draft preparation: Kuljanov Utkir, Kuljanov Jovlon; Writing - review and editing: Zabidin Salleh; Funding acquisition: Zabidin Salleh; Resources: Zabidin Salleh; Validation and Visualisation: Kuljanov Utkir, Zabidin Salleh; Supervision: Zabidin Salleh.

Conflict of Interest Statement

The authors declare that they have no conflict of interest.

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