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BEST PROXIMITY POINT RESULTS FOR FUZZY Z-PROXIMAL CONTRACTIONS IN FUZZY METRIC SPACES

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ABSTRACT

This article explores the formulation of fuzzy Z-proximal contractions within the framework of fuzzy metric spaces. Additionally, it unveils novel findings regarding the existence and singularity of ideal proximity points for such contractive mappings with complete fuzzy metric spaces. These new findings broaden and augment certain results in the existing literature. Examples are provided to support the findings.

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Introduction

Fixed point theory primarily investigates the presence and distinctiveness of points that remain unchanged under transformation, given various contractive conditions and mappings. Specifically, the main goal is to find solutions to operator equations $Tx = x$ for $x \in X$, where $T: X \longrightarrow X$ and X is an abstract space. These solutions are recognised as fixed points of T. Consequently, fixed point theory has vast applications in real-world problems, such as dynamic market equilibrium [1], colour image filtering [2], solving equations of motion [3], and analysing the deformation of an elastic beam [4]. Numerous studies on this theory can be found in the literature [2, 5-8].

By increasing the difficulty of the problem, such as considering the mapping $T: A \longrightarrow B$, where A and B are non-empty subsets of X, the solution to $Tx = x$ might not exist. In this case, the optimal solution, termed the best proximity point, is considered by selecting the point that yields the smallest distance between x and Tx. Some works related to best proximity point theory can be found in [9-12] and the references therein.

In 1942, Menger [13] proposed probabilistic metric spaces, which evaluate the uncertainty regarding distance between points based on the postulation of randomness. After Zadeh [14] introduced the idea of fuzzy sets in 1965, researchers discovered that the uncertainty in distance, in certain cases, is caused by fuzziness rather than randomness. Motivated by this observation, in 1975, Kramosil and Michálek [15] made the first approach and presented fuzzy metric spaces as an augmentation for metric spaces by incorporating the concept of fuzzy sets. Following this, George and Veeramani [16] modified the notion of fuzzy metric spaces to allow the study of Hausdorff topology within this context.

Recently, researchers have shown strong enthusiasm for expanding best proximity point findings to fuzzy metric spaces and their generalisations. Notably, Vetro and Salimi [17] demonstrated several best proximity results for non-self mappings through assorted proximal contractions. Saleem *et al*. [7] explored several optimal coincidence point problems within the framework of non-Archimedean fuzzy metric spaces. Moreover, Saleem *et al*. [7] generalised both fuzzy η-proximal contractions and fuzzy ψ-proximal contractions with admissible property and derived some optimal coincidence point results on fuzzy b-metric spaces. Additionally, Wong *et al*. [18] presented a new Ćirić-type proximal contractive mapping and derived some ideal coincidence results in fuzzy metric spaces.

Drawing inspiration from the works of Shukla *et al*. [19] and Gopal *et al*. [20], the fuzzy Z contractive condition is extended to the proximal setting. Fuzzy Z-proximal contractions are introduced, followed by the validation of several best proximity point results concerning mappings that satisfy such proximal contractive conditions.

Preliminaries

This section presents the fundamental definitions and concepts used throughout the article.

Definition 2.1 [21]. A binary operation $*$: $[0,1]$ x $[0,1] \longrightarrow [0,1]$ is referred to as a continuous t-norm if the following conditions are satisfied:

i. $a * 1 = a$ for every a in [0,1];

ii. $*$ is associative and commutative;

iii. $a * b \leq i * j$ provided $a \leq i$ and $b \leq j$, where $a, i, b, j \in [0,1]$; and,

iv. $*$ is continuous.

Definition 2.2 [16]. A triplet $(X, M, *)$ is called a fuzzy metric space whenever X is arbitrary non-empty set, $*$ denotes a continuous t-norm, and M represents a fuzzy metric defined on X x X x $(0, \infty)$, satisfying the subsequent conditions for every $u, v, w \in X$, and $t, s > 0$:

i. $M(u, v, t) > 0$;

ii. $M(u, v, t) = 1$ for all $t > 0$ if, and only if, $u = v$;

iii. $M(u, v, t) = M(v, u, t)$;

iv. $M(u, w, t + s) \geq M(u, v, t) * M(v, w, s)$; and,

v. $M(u, v, \cdot) : (0, \infty) \longrightarrow (0,1]$ is continuous.

As indicated in the previous definition, it is emphasised that throughout the article, the interpretation of fuzzy metric spaces follows that of George and Veeramani. Note that 0 is not considered a valid value for the fuzzy metric M, which aligns with the classical metric scenario, where the value ∞ is not considered.

Lemma 2.1 [22]. Consider a fuzzy metric space $(X, M, *)$. Then, for any $u, v, \in X$, the fuzzy set M $(u, v,\cdot)$ is non-decreasing on $(0, \infty)$.

Definition 2.3 [16]. Consider a fuzzy metric space $(X, M, *)$. A sequence $\{u_n\} \subseteq X$ is:

i. Convergent to $v \in X$ if, for any $\epsilon \in (0, 1)$ and $t > 0$, there exists $u_0 \in \mathbb{N}$ such that $M\ (u_n, v, t) > 1 - \epsilon$ for all $n \geq n_0$, i.e., $M\ (u_n, v, t) \to 1$ as $n \to \infty$ for all $t > 0$;

ii. Cauchy if, for all $\epsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M\ (u_n, u_m, t) > 1 - \epsilon$ for any $n, m \geq n_0$.

Definition 2.4 [16]. A fuzzy metric space $(X, M, *)$ is complete if every Cauchy sequence in X converges.

Definition 2.5 [17]. Let A and B denote non-empty subsets of a fuzzy metric space $(X, M, *)$. For all $t > 0$, the sets $A_0(t)$ and $B_0(t)$ are defined as:

$$A_0(t) = \{u \in A \colon M(u, v, t) = M(A, B, t) \text{ for some } u \in B\},$$

$$B_0(t) = \{v \in B \colon M(u, v, t) = M(A, B, t) \text{ for some } u \in A\}$$

where:

$$M(u, v, t) = \sup \{M(A, B, t) \text{ for } a \in A \text{ and } b \in B\}$$

for all $t > 0$. Furthermore, the distance between a point $u \in X$ and the non-empty set A is given by $M\ (u, A, t) = \sup_{a \in A} M\ (u, a, t)$ for all $t > 0$.

For two non-empty subsets A and B of a fuzzy metric space $(X, M, *)$ and a mapping $T{:}A \longrightarrow B$, a point $u \in A$ is said to be a best proximity point of T if $M\ (u, Tu, t) = M\ (A, B, t)$ hold for every $t > 0$.

Definition 2.6 [23]. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$. The set B is considered fuzzy approximately compact in relation to A if all sequence $\{v_n\} \subseteq B$ and specific u $\in A$ such that $M\ (u, v_n, t) \to M\ (u, B, t)$ and $u \dotplus\epsilon\ A_0(t)$ holds.

Definition 2.7 [19]. The set Z comprises all mapping $\zeta : (0, 1] \times (0, 1]$ into $\mathbb{R}$ that satisfy the following condition:

$$\zeta\ (t, s) > s \text{ for each } t, s \in (0, 1).$$

From Definition 2.7, it is clear that $\zeta\ (t, t) > t$ for any $t \in (0, 1)$.

Example 2.1 [19]. Suppose the mappings $\zeta : (0, 1] \times (0, 1] \to \mathbb{R}$ are described as follows:

i. $\zeta\ (t, s) = \frac{t}{s}$;

ii. $\zeta\ (t, s) = 1 + k\ (s - 1)$ where $k \in (0, 1)$; and,

iii. $\zeta(t, s) = \begin{cases} t, & t > s; \\ \sqrt{2}, & \text{otherwise.} \end{cases}$

Consequently, in each case, $\zeta \in Z$.

Definition 2.8 [19]. Consider a fuzzy metric space $(X, M, *)$ and a mapping $T{:}X \longrightarrow X$. Suppose there exists $\zeta \in Z$ such that:

$$M\ (Tu, Tv, t) \geq \zeta\ \big(M\ (Tu, Tv, t), M\ (u, v, t)\big)$$

for all $u, v \in X, Tu \neq Tv$, and $t > 0$. Then T is referred to as a fuzzy Z-contraction in relation to the mapping $\zeta \in Z$.

Main Results

This section begins with the following definition:

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space, and let $T{:}A \longrightarrow B$ be a mapping where A and B are non-empty subsets of X. Suppose there exists $\zeta \in Z$ such that

$$\left.\begin{aligned} M(u, Tx, t) &= M(A, B, t) \\ M(v, Ty, t) &= M(A, B, t) \end{aligned}\right\} \Longrightarrow M(u, v, t) \geq \zeta\big(M(u, v, t), M(x, y, t)\big) \tag{1}$$

for each $u, v, x, y\ \in A$ and $t > 0$. Then, mapping T is referred to as a fuzzy Z-proximal contractive mapping in relation to $\zeta \in Z$.

The following property is used to prove the main results.

Definition 3.2. Let A and B be non-empty subsets of a fuzzy metric space $(X, M, *)$, and let $T{:}A \longrightarrow B$ be a non-self mapping with $\zeta \in Z$. The quadruple (X, M, T, ζ) is said to possess property (SP) if for any sequence $\{u_n\}$ starting at $u_0 \in A$ and $u_n\ (t) \in A_0\ (t)$ for all $n \in \mathbb{N}$, the following condition holds:

$$\inf_{m>n} M(u_{n+1}, u_{m+1}, t) \geq \inf_{m>n} M(u_n, u_m, t)$$

for all $n \in \mathbb{N}$ and $t > 0$ implies that:

$$\lim_{n\to\infty} \inf_{m>n} \zeta\big(M(u_{n+1}, u_{m+1}, t), M(u_n, u_m, t)\big) = 1 \text{ for any } t > 0.$$

Remark 3.1. In the previous definition, property (*SP*) is defined for non-self mappings, where the label "*SP*" is used merely for identification. In Shukla *et al*. [19], a version of this property for self-mappings was established, denoted as property (*S*).

Next, the article's main results are presented.

Theorem 3.1. Let $(X, M, *)$ be a complete fuzzy metric space. Consider non-empty subsets A and B of X, where B is fuzzy approximately compact in relation to A. Suppose that $T{:}A \longrightarrow B$ is a fuzzy Z-proximal contractive mapping satisfying the following conditions:

i. $A_0\ (t)$ is non-empty, and $\big(T\ A_0\ (t)\big) \subseteq B_0\ (t)$; and,
ii. The quadruple (X, M, T, ζ) possess property (SP).

Then, the mapping T yields a unique best proximity point.

Proof. Select an arbitrary point $u_0 \in A_0\ (t)$ as $A_0\ (t)$ is non-empty. Since $T\ (u_0) \in T\ \big(A_0\ (t)\big) \subseteq B_0\ (t)$, there exists $u_1 \in A_0\ (t)$ such that:

$$M(u_1, Tu_0, t) = M(A, B, t) \text{ for all } t > 0.$$

Similarly, since $T(u_1) \in T\left(A_0(t)\right) \subseteq B_0(t)$, there exists $u_2 \in A_0(t)$ such that:

$$M(u_2, Tu_1, t) = M(A, B, t) \text{ for all } t > 0.$$

Repeating this process, a sequence $\{u_n\}$ in $A_0(t)$ is obtained, satisfying:

$$M(u_{n+1}, Tu_n, t) = M(A, B, t)$$

for each $n \in \mathbb{N} \cup \{0\}$ and $t > 0$.

If $u_n = u_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then it follows $M(u_n, Tu_n, t) = M(u_{n+1}, Tu_n, t) = M(A, B, t)$ which confirms that u_n is the optimal proximity point of T. Now, assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. For all $t > 0$, since:

$$M(u_{n+1}, Tu_n, t) = M(A, B, t),$$
$$M(u_{n+2}, Tu_{n+1}, t) = M(A, B, t),$$

by (1) and the feature of mapping ζ, it can be deduced that:

$$\begin{aligned} M(u_{n+1}, u_{n+2}, t) &\geq \zeta\left(M(u_{n+1}, u_{n+2}, t), M(u_n, u_{n+1}, t)\right) \\ &> M(u_n, u_{n+1}, t) \end{aligned}$$

which leads to:

$$M(u_{n+1}, u_{n+2}, t) > M(u_n, u_{n+1}, t)$$

for each $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Consider $u_m = u_n$ for some $n < m$. For each $t > 0$, observe that:

$$M(u_{m+1}, Tu_m, t) = M(A, B, t),$$
$$M(u_{n+1}, Tu_n, t) = M(A, B, t).$$

By (1) and the feature of mapping ζ, it follows that:

$$\begin{aligned} M(u_{m+1}, u_{n+1}, t) &\geq \zeta\left(M(u_{m+1}, u_{n+1}, t), M(u_m, u_n, t)\right) \\ &> M(u_m, u_n, t) \\ &= 1 \end{aligned}$$

which is not true. Hence, assume that $u_m \neq u_n$ for all distinct $m, n \in \cup\, \mathbb{N}$. Define, for each $t > 0$,

$$a_n(t) = \inf_{m>n} M(u_n, u_m, t).$$

Taking the infimum for $m > n$ in the preceding inequality results in:

$$\inf_{m>n} M(u_{m+1}, u_{n+1}, t) \geq \inf_{m>n} \zeta\left(M(u_{m+1}, u_{n+1}, t), M(u_m, u_n, t)\right) \geq \inf_{m>n} M(u_m, u_n, t) \tag{2}$$

which implies:

$$\inf_{m>n} M(u_{m+1}, u_{n+1}, t) \geq \inf_{m>n} M(u_m, u_n, t)$$
$$a_{n+1}(t) \geq a_n(t)$$

for all $n \in \mathbb{N} \cup \{0\}$. Since $\{a_n(t)\}$ is monotonic and bounded by (0,1] for each $t > 0$, it follows that the limit of $a_n(t)$ exists for each $t > 0$, that is, $\lim_{n\to\infty} a_n(t) = a(t)$, where $a(t) \in (0,1]$ and $t > 0$.

Next, it is claimed that $a(t) = 1$ for all $t > 0$. Suppose there exist $s > 0$ such that $a(s) < 1$. Since the quadruple (X, M, T, ζ) possesses property (*SP*), it follows that:

$$\lim_{n\to\infty} \inf_{m>n} \zeta\big(M(u_{m+1}, u_{n+1}, s), M(u_m, u_n, s)\big) = 1. \tag{3}$$

From (2), one deduces:

$$\inf_{m>n} M(u_{m+1}, u_{n+1}, s) \geq \inf_{m>n} \zeta\big(M(u_{m+1}, u_{n+1}, t), M(u_m, u_n, s)\big) \geq \inf_{m>n} M(u_m, u_n, s)$$

which can be rewritten as:

$$a_{n+1}(s) \geq \inf_{m>n} \zeta\big(M(u_{m+1}, u_{n+1}, t), M(u_m, u_n, s)\big) \geq a_n(s).$$

Now, taking $n \longrightarrow \infty$ in the preceding inequality, along with (3), it follows that:

$$\lim_{n\to\infty} \inf_{m>n} M(u_n, u_m, s) = a(s) = 1,$$

which contradicts the assumption. As a result, $a(t) = 1$ for all $t > 0$, and by the definition of $a_n(t)$, it follows that $\lim_{n\to\infty} M(u_m, u_n, t) = 1$ for every $t > 0$. Therefore, $\{u_n\}$ behaves as a Cauchy sequence within the closed subset A.

Considering that $(X, M, *)$ is complete, there exists $u^* \in A$ such that $\lim_{n\to\infty} M(u_n, u^*, t) = 1$ for all $t > 0$. Furthermore,

$$M(u_{n+1}, B, t) \geq M(u_{n+1}, Tu_n, t) = M(A, B, t) \geq \mathrm{M}(\mathrm{u}_{n+1}, \mathrm{B}, \mathrm{t}).$$

As $u_n \longrightarrow u^*$, taking $n \longrightarrow \infty$ in the preceding inequality leads to $M(u^*, Tu_n, t) \to M(u^*, B, t)$ for each $t > 0$. Since $T(u_n) \subset B$ and B is fuzzy approximately compact in relation to A, it follows that $u^* \in A_0\,(t)$. Furthermore, since $T(u^*) \in T\big(A_0(t)\big) \subseteq B_0(t)$, it implies that there exists $w \in A_0(t)$ such that:

$$M(w, Tu^*, t) = M(A, B, t) \quad \text{for each } t > 0. \tag{4}$$

It is claimed that $w = u^*$. From $M(w, Tu^*, t) = M(A, B, t)$ and $M(u_{n+1}, Tu_n, t) = M(A, B, t)$, by (1) and the property of mapping ζ, it follows that:

$$\begin{aligned} M(w, u_{n+1}, t) &\geq \zeta\big(M(w, u_{n+1}, t), M(u^*, u_n, t)\big) \\ &> M(u^*, u_n, t). \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$ in the preceding inequality gives $M(w, u^*, t) = 1$ for each $t > 0$, or equivalently, $w = u^*$. Consequently, from (4), it follows that:

$$M(u^*, Tu^*, t) = M(A, B, t) \quad \text{for all } t > 0,$$

which concludes the establishment of the best proximity point for the mapping T.

To establish the uniqueness, assume u^* and v^* are distinct best proximity points of the mapping T. There exists $s > 0$ such that $1 > M(u^*, v^*, s)$. Since:

$$M(u, Tu^*, t) = M(A, B, t),$$
$$M(v^*, Tv^*, t) = M(A, B, t),$$

by (1) and the property of mapping ζ, it follows that:

$$M(u^*,v^*,s) \geq \zeta\big(M(u^*,v^*,s), M(u^*,v^*,s)\big)$$
$$> M(u^*,v^*,s)$$

which leads to a contradiction. Thus, u^* and v^* are identical, confirming the uniqueness property.

An example is presented to illustrate obtained main result.

Example 3.1. Consider $X = \mathbb{N}$ and let the fuzzy metric $M : X \times X \times (0,\infty) \to (0,1]$ be defined by:

$$M(u,v,t) = \frac{\min\{u,v\}}{\max\{u,v\}}$$

for all $t > 0$. Subsequently, $(X,M,*)$ forms a complete fuzzy metric space, where $*$ represents a continuous t-norm on $[0,1] \times [0,1]$ defined by $a * b = ab$ for any $a, b \in [0,1]$. Let $A = \{1,3,5,6,9\}$ and $B = \{2,4,6,8,10\}$. It can be easily verified that A and B are closed subsets of X, and both are non-empty. Furthermore, B is fuzzy approximately compact in relation to A. For any $t > 0$,

$$M(A,B,t) = \frac{9}{10}.$$

It is evident that $A_0(t) = \{9\}$ and $B_0(t) = \{10\}$ for each $t > 0$. Define mapping $T: A \to B$ by

$$T(u) = \begin{cases} 10, & u = 9; \\ u+3, & \text{otherwise} \end{cases} \qquad \text{for all } u \in A.$$

Additionally, the mapping $\zeta: (0,1] \times (0,1] \to \mathbb{R}$ is expressed as:

$$\zeta(t,s) = \begin{cases} \dfrac{t+s}{2}, & t > s; \\ 1, & \text{otherwise} \end{cases} \qquad \text{for all } t, s \in (0,1].$$

Observe that $T\big(A_0(t)\big) \subseteq B_0(t)$,, and the quadruple (X,M,T,ζ) possesses property (*SP*).

Suppose $M(u,Tv,t) = M(A,B,t)$ holds for some $u, v \in A$. The pairs $(u,v) = (9,7)$ and $(u,v) = (9,9)$ satisfy the assumption. Since $M(9,T(9),t) = M(A,B,t)$ and $M(9,\ T(7),t) = M(A,B,t)$, from (1), it is claimed that:

$$M(9,9,t) \geq \zeta\big(M(9,9,t), M(7,9,t)\big)$$

for any $t > 0$. Indeed, $M(9,9,t) = 1$ and $M(7,9,t) = 7/9$ for all $t > 0$. Based on the construction of the mapping ζ, it follows that:

$$\zeta\big(M(9,9,t), M(7,9,t)\big) = \frac{1+\frac{7}{9}}{2}$$
$$= \frac{8}{9}$$

which leads to:

$$1 = M(9,9,t) \geq \zeta\big(M(9,9,t), M(7,9,t)\big) = \frac{8}{9}.$$

This verifies the claim and demonstrates that the mapping T functions as a fuzzy Z-proximal contractive mapping in relation to ζ. Accordingly, all preconditions of Theorem 1 are fulfilled, which implies the existence and uniqueness of the best proximity point of T. Specifically, 9 stands as the sole best proximity point of T.

Shukla *et al*. [19] have shown that fuzzy ψ-contractive mappings imply fuzzy Z-contractive mappings, though the converse does not inevitably hold. This setting remains valid for proximal contractions, which leads to the following definition and corollary. In the following, Ψ is denoted the collection of all mappings ψ: $(0, 1] \longrightarrow (0, 1]$, where ψ is non-decreasing, continuous, and $\psi(t) > t$ for any $t \in (0, 1)$.

Definition 3.3. Let $(X, M, *)$ be a fuzzy metric space and T: $A \to B$ be a mapping where A, B are non-empty subsets of X. Assume there exists $\psi \in \Psi$ which satisfies

$$\left.\begin{array}{l} M(u,Tx,t) = M(A,B,t) \\ M(v,Ty,t) = M(A,B,t) \end{array}\right\} \Rightarrow M(u,v,t) \geq \psi\big(M(x,y,t)\big) \tag{5}$$

for any $u, v, x, y \in A$ and $t > 0$. Then, the mapping T is termed as fuzzy ψ-proximal contractive mapping.

Corollary 3.1. Let $(X, M, *)$ be a complete fuzzy metric space. Consider A, B as non-empty subsets of X, where B is fuzzy approximately compact in relation to A. Assume that T: $A \to B$ is a fuzzy ψ-proximal contractive mapping such that the following conditions hold:

i. $A_0(t)$ is non-empty and $T\big(A_0(t)\big) \subseteq B_0(t)$; and,
ii. The quadruple (X, M, T, ζ) possesses property (SP).

Then, the mapping T yields a unique best proximity point.

Proof. The proof follows from Theorem 3.1 by letting $\zeta(t, s) = \psi(s)$ for every $t, s \in (0, 1]$.

The following example illustrates that fuzzy Z-proximal contractive mapping is broader than fuzzy ψ-proximal contractive mapping.

Example 3.2. Let $X = \mathbb{R}$ and a fuzzy metric $M : X \times X \times (0, \infty) \to (0, 1]$ be defined by

$$M(u, v, t) = \frac{t}{t + |u - v|}$$

for all $t > 0$. Consequently, $(X, M, *)$ forms a complete fuzzy metric space, with $*$ representing a continuous t-norm on [0, 1] ×[0,1] by $a * b = ab$ for any $a, b \in [0, 1]$. Consider $A, B \subseteq X$ defined as $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6\}$. It is straightforward to check that A and B are closed subsets of X that are non-empty, and that B is fuzzy approximately compact in relation to A. Additionally, for any $t > 0$,

$$M(A, B, t) = 1$$

With some calculation, it can be established that $A_0(t) = B_0(t) = \{1, 2, 3, 4, 5\}$ for each $t > 0$. Define the mappings $T: A \to B$ and $\zeta: (0, 1] \times (0, 1] \to \mathbb{R}$ as follows:

$$T(u) = \begin{cases} u + 1, & u \in \{1, 2, 3\}; \\ 5, & \text{otherwise} \end{cases} \qquad \text{for all } u \in A$$

and

$$\zeta(t,s) = \begin{cases} \frac{t+s}{2}, & t > s; \\ 1, & \text{otherwise} \end{cases} \quad \text{for all } t,s \in (0,1].$$

Observe that $T(A_0(t)) \subseteq B_0(t)$, while the quadruple (X, M, T, ζ) possesses property (*SP*). One can verify that the mapping T functions as fuzzy Z-proximal contractive mapping in relation to ζ. Accordingly, all preconditions of Theorem 1 are fulfilled, which imply the existence and uniqueness of the best proximity point of T. Specifically, 5 stands as the sole best proximity point of T.

In contrast, mapping T does not function as a fuzzy ψ-proximal contractive mapping. Indeed, let $\psi(t) = \sqrt{t}$ for every $t \in (0,1]$. Clearly, ψ belongs to the class Ψ. It holds that

$$M(4, T(3), t) = M(A, B, t) \text{ and } M(5, T(5), t) = M(A, B, t) \quad \text{for all } t > 0.$$

Consider $t = 1$. This yields

$$M(4,5,t) = 0.5 \text{ and } \psi\big(M(3,5,1)\big) = \sqrt{\frac{1}{3}} = 0.57734$$

Thus,

$$M(4,5,t) \geq \psi\big(M(3,5,t)\big)$$

for all $t > 0$. Therefore, mapping T does not satisfy (5), demonstrating that T is not a fuzzy ψ-proximal contractive mapping.

Conclusions

This article introduces the concept of fuzzy Z-proximal contractions in fuzzy metric spaces, establishing the existence and uniqueness of best proximity points for such proximal contractions. Examples are provided to illustrate the results. It is important to note that by restricting mappings to self-mappings, fixed-point results, as presented in [19], are obtained.

Utilising the admissible property on fuzzy ψ-proximal contractions, Saleem *et al*. [24] defined fuzzy $(\beta - \psi)$-generalised proximal contractions in fuzzy b-metric spaces. It would be intriguing to apply the admissible property to fuzzy Z-proximal contractions and investigate their relationship with fuzzy $(\beta - \psi)$-generalised proximal contractions. Further, expanding the concept of fuzzy Z-proximal contractions by incorporating various approaches, such as the Suzuki type and interpolative contractive conditions, as demonstrated in [20] and [25], offers potential for further development. As evidenced by Younis *et al*. [26], applying the obtained results in various engineering science-related challenges would also be valuable.

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Conflicts of Interest Statement

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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