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ON THE GENERALISED SECOND-ORDER LINEAR RECURRENCES RELATIONS AND IDENTITIES

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ABSTRACT

In this article, we study the sequence $\{V_n\}$, which is generated by the (p,q)-Generalised linear recurrence relation of second order $V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}, n \geq 2$, with the initial terms $V_0 = a, V_1 = b$, where p, q, a and b $(ab \neq 0, pq \neq 0)$ are arbitrary real numbers. Addition, subtraction formulas, the Binet formula and some new results were obtained and studied in the generalised form. Some existing and new identities are also explored, employing this generalised definition of the sequence $\{V_n\}$ and becoming the special cases on substituting the coefficients p, q of the recurrence relation and the initial terms V_0, V_1.

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Introduction

Sequences initiated from linear recurrences are the fabulous numerical numbers in mathematical sciences; authors [1, 2, 3] studied the second-order sequences extensively and have various significant applications in fields such as physical, computational, and applied sciences to work out the solutions of numerous combinatorial and data structure problems. Author [5] investigated the generalised relations between the Fibonacci for the coding theory application. Researchers [8, 9, 10, 11, 15] exposed power formulas, sums, and properties of Fibonacci numbers through binomial coefficients and matrix methods.

Researchers [6, 7, 14] studied and derived numerous closed-form formulas and identities for second-order linear recurrences. Authors [4, 16, 17, 20, 21, 22] studied several well-known fascinating identities and used the matrix representations of the Fibonacci sequences to study the sequence properties and investigate the power of this form of representation. Vajda [18] expressed identities concerning generalised Fibonacci numbers and binomial coefficients. For the applications of Fibonacci numbers and their relationships (see [7, 12, 17]) their occurrences in nature.

In this paper, second-order linear recurrences are expressed in the most (p,q) generalised form to obtain Fibonacci-like numbers in the general formula. The generating function and Binet's formula are also obtained in the generalised form. Some existing formulas are found to be comprehensive, involving the initial terms and the coefficients p and q arbitrarily. It is shown that various existing results and identities become the special cases of the obtained results with regard to the generalised definition stated in this article.

Definition 1. For any arbitrary real numbers p, q, a and b, we define the Fibonacci numbers $V_n\left(p,q,a,b\right)$ generated by the generalised homogeneous recursion relation as

$$V_n\left(p,q,a,b\right) = pV_{n-1} + qV_{n-2}\ ,\ n \geq 2,\ V_0 = a,\ V_1 = b. \tag{1}$$

The numbers $V_n\left(p,q,V_1,V_2\right)$ defined in (1) can also be obtained using

$$V_n\left(p,q,a,b\right) = T_{n-1}\left(p,q\right)b + qT_{n-2}(p,q)a,\ n \geq 2$$

where

$$\begin{pmatrix} T_n \\ T_{n-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, n > 2, T_0, = 0, T_1 = p, T_2 = 1.$$

The above sequence is called the generalised $(p, q,)$ -Fibonacci sequence and will be denoted in short by $\{V_n\}$.

First Ten Terms of $(p, q,)$ ***- Generalised Sequence***

Table 1: First ten terms of the generalised V_n (p, q, V_0, V_1) sequence

Sequence terms (n, V_n)	**V_n (p, q, V_0, V_1)**
$n = 0$, V_0	V_0
$n = 1$, V_1	V_1
$n = 2$, V_2	$qV_0 + pV_1$
$n = 3$, V_3	$qpV_0 + (p^2 + q)V_1$
$n = 4$, V_4	$q(p^2 + q)V_0 + (p^3 + 2pq)V_1$
$n = 5$, V_5	$q(p^3 + 2pq)V_0 + (p^4 + 3p^2q + q^2)V_1$
$n = 6$, V_6	$q(p^4 + 3p^2q + q^2)V_0 + (p^5 + 4p^3q + 3pq^2)V_1$
$n = 7$, V_7	$q(p^5 + 4p^3q + 3pq^2)V_0 + (p^6 + 5p^4q + 6p^2q^2 + q^3)V_1$
$n = 8$, V_8	$q(p^6 + 5p^4q + 6p^2q^2 + q^3)V_0 + (p^7 + 6p^5q + 10p^3q^2 + 4pq^3)V_1$
$n = 9$, V_9	$q(p^7 + 6p^5q + 10p^3q^2 + 4pq^3)V_0 + (p^8 + 7p^6q + 15p^4q^2 + 10p^2q^3 + q^4)V_1$

Special Cases

When the initial values are $V_1 = 0$, $V_2 = 1$ $p = q = 1$ set, the resulting sequence is the *classical Fibonacci sequence.*

- When $V_0 = 2$, $V_1 = 1$ and $p = q = 1$ then, the (p,q) sequence is the *Lucas sequence.*
- When $V_0 = 2$, $V_1 = 3$ $p = 5$, $q = 7$, the first few numbers of the $\{V_n\}$ are:

  $\{V_n\} = \{2,3,29,166,1033,6327,38866,238619,1465157,8996118,\cdots\}$
- When $V_0 = \frac{1}{2}, V_1 = 3, p = \frac{1}{3}, q = \frac{2}{5}$, the first few numbers of the are:

$$\{V_n\} = \left\{\frac{1}{2}, 3, \frac{6}{5}, \frac{8}{5}, \frac{76}{75}, \frac{44}{55}, \frac{2468}{3375}, \frac{6428}{10125}, \frac{76564}{151875}, \frac{192268}{355625}, \cdots\right\}$$

2.2 Generalised Generating Functions

The generating functions of the sequences $V_n = pV_{n-1} + qV_{n-2}$, $n \geq 2$, $V_0 = a, V_1 = b$, a, b, p and q are any non – zero real numbers, is [Verma [19] equation (5)],

$$\sum_{n=0}^{\infty} V_n x^n = \frac{a+(b-pa)x}{\left(1-px-qx^2\right)}. \tag{2}$$

The generalised Binet form

The (p, q)-generalised numbers can be retrieved through the Binet-like formula [Verma [19] equation (5)],

$$V_n = \left(\frac{a^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)V_0 + \left(\frac{a^n-\beta^n}{\alpha-\beta}\right)\left(V_1 - pV_0\right), \tag{3}$$

where $\alpha, \beta = \dfrac{p \pm \sqrt{p^2-4q}}{2}$ are the roots of $x^2 - px - q = 0$.

Fundamental Identities and Their Generalisation

Theorem 1. (The Binet Form) If α and β are the roots of the characteristic equation, $x^2 - px - q = 0$ then $V_n = A\alpha^n + B\beta^n$ where

$$A = \frac{V_1 - \beta V_0}{\alpha - \beta}, B = \frac{\alpha V_0 - V_1}{\alpha - \beta} \text{ and } \alpha^n = \frac{V_{n+1} - \beta V_n}{V_1 - \beta V_0}, \beta^n = \frac{V_{n+1} - \alpha V_n}{\alpha V_0 - V_1}$$

Since $D = p^2 + 2q \neq 0$ this implies $\alpha \neq \beta$, $\alpha - \beta = \sqrt{p^2 + 4q}$,

$$AB = \frac{F}{D}, F = qV_0^2 + pV_0V_1 - V_1^2,\ V_{n+m} = \left(A\alpha^m\right)\alpha^n + \left(B\beta^m\right)\beta^n.$$

The definition is used for V_n postulates, proofs, and verifications of some existing identities, and after that, more refined procedures are introduced. Rabinowitz [14] described several similar formulas.

Theorem 2. (The Addition Formula for V_n)

For all integers n and m,

$$V_{n+m} = -\frac{1}{\left(V_0V_2 - V_1^2\right)}\begin{vmatrix} V_0 & V_1 & V_m \\ V_1 & V_2 & V_{m+1} \\ V_n & V_{n+1} & 0 \end{vmatrix}.$$

$$V_{n+m} = \frac{\left(V_1V_{m+1} - V_2V_m\right)V_n - \left(V_1V_m - V_0V_{m+1}\right)V_{n+1}}{-\left(V_0V_2 - V_1^2\right)}$$

Horadam [10] described several such formulas.

Verification of theorem (3.2) using the (p, q) generalised definition in (1)

Let $n = 4$, $m = 2$, (V_{n+m} is the V_6)

LHS $= V_{n+m} = \left(p^4q + 3p^2q^2 + q^3\right)V_0 + \left(p^5 + 4p^3q + 3pq^2\right)V_1 = V_6$

RHS $= -\dfrac{1}{\left(V_0V_2 - V_1^2\right)}\begin{vmatrix} V_0 & V_1 & V_m \\ V_1 & V_2 & V_{m+1} \\ V_n & V_{n+1} & 0 \end{vmatrix} = \dfrac{\left[\left(p^4q + 3p^2q^2 + q^3\right)V_0 + \left(p^5 + 4p^3q + 3pq^2\right)V_1\right]\left(qV_0^2 + pV_0V_1 - V_1^2\right)}{\left(qV_0^2 + pV_0V_1 - V_1^2\right)}$

$$= \left(p^4 q + 3p^2 q^2 + q^3\right)V_0 + \left(p^5 + 4p^3 q + 3pq^2\right)V_1 = V_6.$$

Hence, the result is verified.

(*p, q*)-Generalised Sequence using Binomial Coefficients

Theorem 3. (The Addition Formula for V_n)

$$V_{n+m} = \sum_{k=0}^{m} \binom{m}{k} V_{n-k} p^{m-k} q^k \ , \ n-k>0.$$

Proof. Mathematical Induction is used to check the validity of the result

$$V_{n+m} = \sum_{k=0}^{m} \binom{m}{k} V_{n-k} p^{m-k} q^k \ , \ n-k>0,$$

When $m = 1$,

$$\text{LHS} = V_{n+1},$$

$$\text{RHS} = \sum_{k=0}^{1} \binom{1}{k} V_{n-k} p^{1-k} q^k = pV_n + qV_{n-1} = V_{n+1}.$$

The theorem holds for all integer n when $m = 1$. Using induction, it will be shown that it holds for every positive integer m.

Let the theorem be true up to the value *of m.* Then,

$$V_{t+m} = \sum_{k=0}^{m} \binom{m}{k} V_{t-k} p^{m-k} q^k = \sum_{k=0}^{m} \binom{m}{k} \left(pV_{t-k-1} + qV_{t-k-2}\right) p^{m-k} q^k$$

$$= \sum_{k=0}^{m} \binom{m}{k} \left(pV_{t-k-1}\right) p^{m-k} q^k + \sum_{k=0}^{m} \binom{m}{k} \left(qV_{t-k-2}\right) p^{m-k} q^k$$

$$= V_{t-1} p^{m+1} q^0 + \sum_{k=1}^{m} \binom{m}{k} pV_{t-k-1}\left(p^{m-k} q^k\right) + \sum_{k=0}^{m-1} \binom{m}{k} \left(qV_{t-k-2}\right) p^{m-k} q^k + V_{t-m-2} p^0 q^{m+1}$$

$$= p^{m+1} V_{t-1} + \sum_{k=1}^{m} \binom{m}{k} pV_{t-k-1}\left(p^{m-k} q^k\right) + \sum_{k=1}^{m} \binom{m}{k-1} \left(qV_{t-k-1}\right) p^{m-(k-1)} q^{k-1} + q^{m+1} V_{t-m-2}$$

$$= p^{m+1} V_{t-1} + \sum_{k=1}^{m} \left(\binom{m}{k} + \binom{m}{k-1}\right) V_{t-k-1} p^{m-k+1} q^k + q^{m+1} V_{t-m-2}$$

$$= p^{m+1} V_{t-1} + \sum_{k=1}^{m} \binom{m+1}{k} V_{t-k-1} p^{m-k+1} q^k + V_{t-m-2} q^{m+1}$$

$$V_{t+m} = \sum_{k=0}^{m+1} \binom{m+1}{k} V_{t-k-1} p^{m-k+1} q^k.$$

Let $t = n + 1$, then

$$V_{n+m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} V_{n-k} p^{m+1-k} q^k.$$

This is again the theorem.

Corollary 1. When m = *n,* then we have

$$V_{2n}=\sum_{k=0}^{n}\binom{n}{k}V_{n-k}p^{n-k}q^{k}\ .$$

Corollary 2. When $m = n+1$, then we have

$$V_{2n+1}=\sum_{k=0}^{n+1}\binom{n+1}{k}V_{n-k}p^{n+1-k}q^{k}\ .$$

The subtraction formula *for* V_n

Melham and Shannon [13] expressed the impure subtraction formula in the following form:

Theorem 4. (The Subtraction Formula) For all non-negative integers $m \geq n$

$$V_{m-n}=\frac{V_mT_{n+1}-V_{m+1}T_n}{(-q)^n}\ ,\quad \begin{pmatrix}T_{n+1}\\ T_n\end{pmatrix}=\begin{pmatrix}p & q\\ 1 & 0\end{pmatrix}^{n-1}\begin{pmatrix}p\\ 1\end{pmatrix}.$$

Proof. We prove this formula by induction n and by verifying that both sides satisfy the same recurrence and initial values.

For $n = 0$ the identity simplifies to

$$V_{m-n}=\frac{V_mT_{n+1}-V_{m+1}T_n}{(-q)^n}$$

$$V_{m-0}=\frac{V_mT_1-V_{m+1}T_0}{(-q)^0}=\frac{V_mT_1-V_{m+1}T_0}{(-q)^0}=V_m\quad \because T_1=1,\ T_0=0,$$

So, identity holds $n = 0$.

For $n = 1$, the identity simplifies to

$$V_{m-1}=\frac{V_mT_2-V_{m+1}T_1}{(-q)^1},$$

this implies

$$-qV_{m-1}=V_mT_2-V_{m+1}T_1\quad \because T_2=p,T_1=1 \text{ and } T_0=0,$$

$$V_{m+1}=pV_m+qV_{m-1},$$

So, identity holds $n = 1$.

Assume the identity holds for n, i.e.,

$$V_{m-n}=\frac{V_mT_{n+1}-V_{m+1}T_n}{(-q)^n}$$

Now we shall show that it holds for $n + 1$,

$$V_{m-(n+1)} = \frac{V_m T_{n+2} - V_{m+1} T_{n+1}}{(-q)^{n+1}}$$

$$= \frac{V_m \left(pT_{n+1} + qT_n\right) - V_{m+1} T_{n+1}}{(-q)^{n+1}} \qquad \because T_{n+2} = pT_{n+1} + qT_n.$$

$$= \frac{pV_m T_{n+1} + qV_m T_n - V_{m+1} T_{n+1}}{(-q)^{n+1}}$$

$$= \frac{qV_m T_n + pV_m T_{n+1} - V_{m+1} T_{n+1}}{(-q)^{n+1}}$$

$$= \frac{qV_m T_n + \left(pV_m - V_{m+1}\right) T_{n+1}}{(-q)^{n+1}}$$

$$= \frac{qV_m T_n + \left(pV_m - \left(pV_m + qV_{m-1}\right)\right) T_{n+1}}{(-q)^{n+1}} \qquad \because V_{m+1} = pV_m + qV_{m-1}$$

$$= -q\frac{V_{m-1} T_{n+1} - V_m T_n}{(-q)^{n+1}}$$

$$= \frac{V_{m-1} T_{n+1} - V_m T_n}{(-q)^{n}}$$

By the induction hypothesis, we have $\frac{V_m T_{n+2} - V_{m+1} T_{n+1}}{(-q)^{n+1}} = V_{m-(n+1)}$

So, the formula holds for $n + 1$ completing the proof using induction.

Verification: Let $m = 8$, $n = 4$ then LHS $V_{8-4} = q\left(p^2 + q\right)V_0 + \left(p^3 + 2pq\right)V_1 = V_4$

$$\text{RHS} = \frac{V_m T_{n+1} - V_{m+1} T_n}{(-q)^n} = \frac{q^4\left[q\left(p^2 + q\right)V_0 + \left(p^3 + 2pq\right)V_1\right]}{q^4} = V_4$$

The theorem is verified.

Theorem 5. For all non-negative integers n,

$$\left(V_0 V_{n+1} - V_1 V_n\right)\left(V_0 V_2 - V_1^2\right)^{-1} = T_{n-1},$$

Proof. Employing the relation defined in definition (1), we have

$$V_n\left(p,q,a,b\right)=V_n=T_{n-1}\left(p,q\right)b+qT_{n-2}(p,q)a,\ n\geq 2,$$

$$\begin{pmatrix}T_{n+1}\\T_n\end{pmatrix}=\begin{pmatrix}p & q\\1 & 0\end{pmatrix}^{n-2}\begin{pmatrix}p\\1\end{pmatrix}$$

$$\text{LHS}=\frac{1}{\left(V_0V_2-V_1^2\right)}\left(V_0V_{n+1}-V_1V_n\right)$$

$$=\frac{1}{\left(V_0V_2-V_1^2\right)}\left[V_0\left(T_n\left(p,q\right)b+qT_{n-1}(p,q)a\right)-V_1\left(T_{n-1}\left(p,q\right)b+qT_{n-2}(p,q)a\right)\right]$$

Since $V_0=a,\ V_1=b,\ V_2=pb+qa\ \ \therefore V_0V_2-V_1^2=pab+qa^2-b^2$

$$=\frac{1}{\left(pab+qa^2-b^2\right)}\left(\left[\left(T_n\left(p,q\right)ba-qT_{n-2}(p,q)ab+qT_{n-1}(p,q)a^2-T_{n-1}\left(p,q\right)b^2\right)\right]\right)$$

$$=\frac{1}{\left(pab+qa^2-b^2\right)}\left(ba\left(T_n\left(p,q\right)-qT_{n-2}(p,q)\right)+T_{n-1}(p,q)\left(qa^2-b^2\right)\right)$$

Now using $T_n\left(p,q\right)-qT_{n-2}(p,q)=pT_{n-1}\left(p,q\right)$

$$=\frac{1}{\left(pab+qa^2-b^2\right)}\left(ba\left(pT_{n-1}\left(p,q\right)\right)+T_{n-1}(p,q)\left(qa^2-b^2\right)\right)$$

$$=\frac{1}{\left(pab+qa^2-b^2\right)}\left(T_{n-1}\left(p,q\right)\left(pba+qa^2-b^2\right)\right)=T_{n-1}\left(p,q\right)=RHS$$

Hence, the theorem proved.

Theorem 6. For all non-negative integers *n*,

$$\frac{\left(pV_0-2V_1\right)V_{n+1}+\left(2qV_0+pV_1\right)V_n}{\left(V_0V_2-V_1^2\right)}=K_n$$

$$K_{n+2}=pK_{n+1}+qK_n, n\geq 0, K_0=2, K_1=p$$

Proof. For $n=0$,

$$\text{LHS}=\frac{\left(pV_0-2V_1\right)V_{n+1}+\left(2qV_0+pV_1\right)V_n}{\left(V_0V_2-V_1^2\right)}=\frac{\left(pV_0-2V_1\right)V_1+\left(2qV_0+pV_1\right)V_0}{\left(V_0V_2-V_1^2\right)}$$

Substitute $V_2=pV_1+qV_0$, we obtain

$$=\frac{2\left(pV_0V_1-V_1^2+qV_0^2\right)}{\left(pV_0V_1-V_1^2+qV_0^2\right)}=2=K_0.$$

So, the first initial condition holds.

Now for $n=1$

LHS $= \dfrac{(pV_0 - 2V_1)V_2 + (2qV_0 + pV_1)V_1}{(V_0V_2 - V_1^2)}$, since $V_2 = pV_1 + qV_0$

$$= \frac{(pV_0 - 2V_1)(pV_1 + qV_0) + (2qV_0 + pV_1)V_1}{(V_0(pV_1 + qV_0) - V_1^2)}$$

$$= \frac{p(pV_0V_1 - V_1^2 + qV_0^2)}{(pV_0V_1 - V_1^2 + qV_0^2)} = p = K_1.$$

Consequently, the second initial condition holds.

We show that $K_{n+2} = pK_{n+1} + qK_n, \ n \geq 0$

Using the definition of K_n, we have

$$K_{n+2} = \frac{(pV_0 - 2V_1)V_{n+3} + (2qV_0 + pV_1)V_{n+2}}{(V_0V_2 - V_1^2)}$$

Now substitute $V_{n+2} = pV_{n+1} + qV_n$ the formula

$$= \frac{(pV_0 - 2V_1)(pV_{n+2} + qV_{n+1}) + (2qV_0 + pV_1)(pV_{n+1} + qV_n)}{(V_0V_2 - V_1^2)}$$

Expanding and rearranging, we obtain

$$= p\left[\frac{(pV_0 - 2V_1)V_{n+2} + (2qV_0 + pV_1)V_{n+1}}{(V_0V_2 - V_1^2)}\right] + q\left[\frac{(pV_0 - 2V_1)V_{n+1} + (2qV_0 + pV_1)V_n}{(V_0V_2 - V_1^2)}\right]$$

$$= pK_{n+1} + qK_n.$$

Hence $K_{n+2} = pK_{n+1} + qK_n$. Thus, the recurrence relation holds.

Theorem 7. (Simson's Formula for): For all integers n,

$$\begin{vmatrix} V_{n-1} & V_n \\ V_n & V_{n+1} \end{vmatrix} = (-q)^{n-1}\left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

or

$$\begin{vmatrix} V_n & V_{n+1} \\ V_{n+1} & V_{n+2} \end{vmatrix} = (-q)^n\left(qV_1^2 + pV_1V_2 - V_2^2\right)$$

Proof. For $n = 1$, we have

$$\begin{vmatrix} V_0 & V_1 \\ V_1 & V_2 \end{vmatrix} = V_0V_2 - V_1^2 = V_0(pV_1 + qV_0) - V_1^2$$

$$= (-q)^0\left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

The theorem is valid for $P(n,r) = \begin{vmatrix} V_{n-r} & V_n \\ V_n & V_{n+r} \end{vmatrix}, n > r.$

Assume that the result is valid for $n = k$

Therefore, $\begin{vmatrix} V_k & V_{k+1} \\ V_{k+1} & V_{k+2} \end{vmatrix} = (-q)^k \left(qV_0^2 + pV_0V_1 - V_1^2\right)$

Now for $n = k + 1$, we have

$$\begin{vmatrix} V_{k+1} & V_{k+2} \\ V_{k+2} & V_{k+3} \end{vmatrix} = V_{k+1}V_{k+3} - V_{k+2}^2$$

$$= V_{k+1}\left(pV_{k+2} + qV_{k+1}\right) - V_{k+2}^2 \quad \because V_{k+3} = pV_{k+2} + qV_{k+1}$$

$$= pV_{k+1}V_{k+2} + qV_{k+1}^2 - V_{k+2}^2$$

$$= V_{k+1}\left(pV_{k+2} - V_{k+2}\right) + qV_{k+1}^2$$

$$= V_{k+1}\left(pV_{k+2} - \left(pV_{k+1} + qV_k\right)\right) + qV_{k+1}^2 \quad \because V_{k+2} = pV_{k+1} + qV_k$$

$$= qV_{k+1}^2 - qV_kV_{k+1} = -q\left(V_kV_{k+1} - V_{k+1}^2\right)$$

$$= -q\begin{vmatrix} V_k & V_{k+1} \\ V_{k+1} & V_{k+2} \end{vmatrix} = -q(-q)^k \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{k+1}\left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

Hence, the result follows by induction.

Theorem 8. (Catalan1s Identity for V_n). For all integers n and r, $n > r$

$$\begin{vmatrix} V_{n-r} & V_n \\ V_n & V_{n+r} \end{vmatrix} = (-q)^{n-r} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$\begin{pmatrix} T_r \\ T_{r-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{r-2} \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

Proof. Let $M(n,r) = \begin{vmatrix} V_{n-r} & V_n \\ V_n & V_{n+r} \end{vmatrix}$, n, r are non-negative integers, $n > r$.

If $n = 1$, then there is no appropriate r.

So we explore $n = 2$ in its place, implies that $r = 1$.

$$M(2,1) = \begin{vmatrix} V_1 & V_2 \\ V_2 & V_3 \end{vmatrix} = V_1V_3 - V_2^2 = V_1\left(pV_2 + qV_1\right) - V_2^2$$

$$= V_1\left(p\left(pV_1 + qV_0\right) + qV_1\right) - \left(pV_1 + qV_0\right)^2$$

Using definition (1) and after simplification we have

$$= (-q)\left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

This can be written as

$$= (-q)^{2-1} T_1^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \quad \because T_1^2 = 1$$

Thus $M(2,1)$ is true.

Now for $n = 3$, then appropriate $r = 1$ and 2, after simplification, we have

$$M(3,1) = (-q)^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) = (-q)^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{3-1} T_1^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \quad \because T_1^2 = 1$$

Thus, $M(3,1)$ is true.

$$P(3,2) = -p^2 q \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{3-2} p^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \because T_2 = p$$

$$= (-q)^{3-2} T_2^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

Thus, $M(3,2)$ is true.

This is our basis for the induction.

If $M(n,r)$ is true for all r, where $n > 3$, then, $M(n+1,r)$ follows logically for all r.

So, the induction hypothesis is

$$\begin{vmatrix} V_{n-r} & V_n \\ V_n & V_{n+r} \end{vmatrix} = (-q)^{n-r} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \ \forall\, n > r$$

We need to show:

$$\begin{vmatrix} V_{n+1-r} & V_{n+1} \\ V_{n+1} & V_{n+1+r} \end{vmatrix} = (-q)^{n+1-r} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \quad \forall\, n > r.$$

Induction step, when $r = 1$, we have

$$\begin{vmatrix} V_n & V_{n+1} \\ V_{n+1} & V_{n+2} \end{vmatrix} = V_n V_{n+2} - V_{n+1}^2$$

$$\begin{aligned}
&= V_n \left(pV_{n+1} + qV_n\right) - V_{n+1}^2 \\
&= pV_n V_{n+1} + qV_n^2 - V_{n+1}^2 \\
&= V_{n+1}\left(pV_n - V_{n+1}\right) + qV_n^2 \\
&= V_{n+1}\left(pV_n - \left(pV_n + qV_{n-1}\right)\right) + qV_n^2 \\
&= -qV_{n-1}V_{n+1} + qV_n^2 \\
&= -q\left(V_{n-1}V_{n+1} - V_n^2\right) \ = (-q)^n \left(qV_0^2 + pV_0V_1 - V_1^2\right)
\end{aligned}$$

Using $\begin{vmatrix} V_{n-1} & V_n \\ V_n & V_{n+1} \end{vmatrix} = \left(V_{n-1}V_{n+1} - V_n^2\right) = (-q)^{n-1} T_1^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \quad \because T_1 = 1$

So, $M(n+1,1)$ holds.

Now we shall show that, if $M(n+1,r)$ is true, for $n > r > 2$, then, $M(n+1,r+1)$ follows logically.

Our 2nd Induction Hypothesis

$$\begin{vmatrix} V_{n-r+1} & V_{n+1} \\ V_{n+1} & V_{n+r+1} \end{vmatrix} = (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \ \forall\ n > r$$

$$V_{n-r+1}V_{n+r+1} - V_{n+1}^2 = (-q)^{n-r+1} T_{r-1}{}^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right).$$

Then, it is required to show that

$$\begin{vmatrix} V_{n-r} & V_{n+1} \\ V_{n+1} & V_{n+r+1} \end{vmatrix} = (-q)^{n-r} T_{r+1}{}^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) \ \forall\ n > r$$

$$\text{LHS} = \begin{vmatrix} V_{n-r} & V_{n+1} \\ V_{n+1} & V_{n+r+2} \end{vmatrix}$$

$$= V_{n-r}V_{n+r+2} - V_{n+1}^2$$

$$= V_{n-r}V_{n+r+2} + (V_{n-r+1}V_{n+r+1} - V_{n+1}^2) - V_{n-r+1}V_{n+r+1}$$

$$= V_{n-r}V_{n+r+2} + (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - V_{n-r+1}V_{n-r+1}$$

$$\because V_{n-r+1}V_{n+r+1} - V_{n+1}^2 = (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + V_{n-r}V_{n+r+2} - V_{n-r+1}V_{n+r+1}$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + V_{n-r}\left(pV_{n+r+1} + qV_{n+r}\right) - V_{n-r+1}V_{n+r+1}$$

$$\because V_{n+r+2} = pV_{n+r+1} + qV_{n+r},$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + V_{n+r+1}\left(pV_{n-r} - V_{n-r+1}\right) + qV_{n-r}V_{n+r}$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + V_{n+r+1}\left(pV_{n-r} - V_{n-r+1}\right) + qV_{n-r}V_{n+r}$$

$$\because V_{n-r+1} = pV_{n-r} + qV_{n-r-1}$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + V_{n+r+1}\left(pV_{n-r} - \left(pV_{n-r} + qV_{n-r-1}\right)\right) + qV_{n-r}V_{n+r}$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - qV_{n-r-1}V_{n+r+1} + qV_{n-r}V_{n+r}$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - qV_{n-r-1}V_{n+r+1} + qV_n^2 + qV_{n-r}V_{n+r} - qV_n^2$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - q(V_{n-r-1}V_{n+r+1} - V_n^2) + q(V_{n-r}V_{n+r} - V_n^2)$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - q(V_{n-r-1}V_{n+r+1} - V_n^2) + q(V_{n-r}V_{n+r} - V_n^2)$$

$$\because V_{n-r-1}V_{n+r+1} - V_n^2 = (-q)^{n-r-1} T_{r+1}{}^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$\because V_{n-r}V_{n+r} - V_n^2 = (-q)^{n-r} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{n-r+1} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) - q(-q)^{n-r-1} T_{r+1}{}^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right) + q(-q)^{n-r} T_r^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right)$$

$$= (-q)^{n-r} \left(qV_0^2 + pV_0V_1 - V_1^2\right)\left[-qT_r{}^2 + T_{r+1}{}^2 + qT_r^2\right]$$

$$= (-q)^{n-r} T_{r+1}{}^2 \left(qV_0^2 + pV_0V_1 - V_1^2\right).$$

Thus, $M(n+1,r) \Rightarrow M(n+1,r+1)$ the result follows by the induction.

Hence, $M(n,r) \Rightarrow M(n+1,r)$ the result follows by the induction.

Hence, the Theorem proved.

Kalman and Mena [10] generalised this result as

Theorem 9. *Horadam generalisation, for integers $n \geq r$,*

$$\begin{vmatrix} V_{n-r} & V_{n+t} \\ V_n & V_{n+r+t} \end{vmatrix} = q^{n-r} T_r T_{r+t} \left(qV_1^2 + pV_1V_2 - V_2^2 \right)$$

$$\begin{pmatrix} T_r \\ T_{r-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{r-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, \begin{pmatrix} T_{r+t} \\ T_{r+t-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{r+t-2} \begin{pmatrix} p \\ 1 \end{pmatrix}$$

Melham and Shannon [13] expressed this as in the following theorem.

Theorem 10. Horadam generalisation

$$\begin{vmatrix} V_{n+r+s} & V_{n+s} \\ V_{n+r} & V_n \end{vmatrix} = (-q)^n T_r T_s \left(qV_1^2 + pV_1V_2 - V_2^2 \right)$$

$$\begin{pmatrix} T_j \\ T_{j-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{j-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, j = r, s.$$

Theorem 11. (D'Ocagme' s Identity for V_n) For all integers n and m Horadam generalisation

$$\begin{vmatrix} V_m & V_n \\ V_{m+1} & V_{n+1} \end{vmatrix} = (-1)^{n-1} q^n T_{m-n} \left(qV_1^2 + pV_1V_2 - V_2^2 \right)$$

$$\begin{pmatrix} T_j \\ T_{j-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{j-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, j = m - n.$$

Theorem 12. (Generalisation D'Ocagme' s Identity for V_n). For all integers n and m, Horadam generalisation

$$\begin{vmatrix} V_m & V_n \\ V_{m+r} & V_{n+r} \end{vmatrix} = (-1)^{n-1} q^n T_{m-n} T_r \left(qV_1^2 + pV_1V_2 - V_2^2 \right)$$

$$\begin{pmatrix} T_j \\ T_{j-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{j-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, j = r, m - n.$$

Theorem 13. *For all integers n,*

$$\begin{vmatrix} V_0 & V_n \\ V_n & V_{2n} \end{vmatrix} = T_n^2 \left(qV_1^2 + pV_1V_2 - V_2^2 \right), \quad \begin{pmatrix} T_j \\ T_{j-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{j-2} \begin{pmatrix} p \\ 1 \end{pmatrix}, j = n.$$

Theorem 14. For all integers m, n, r, s, t, u, and w

$$\begin{vmatrix} V_{m+r} & V_{m+u} & V_{m+v} & V_{m+w} \\ V_{n+r} & V_{n+u} & V_{n+v} & V_{n+w} \\ V_{s+r} & V_{s+u} & V_{s+v} & V_{s+w} \\ V_{t+r} & V_{t+u} & V_{t+v} & V_{t+w} \end{vmatrix} = 0$$

Theorem 15. For V_n and V_{n+1} involving V_{n+s} and V_{n+t} for two distinct integers *s* and *t*.

$$\begin{pmatrix} V_n \\ V_{n+1} \end{pmatrix} = \frac{1}{V_{s+1}V_t - V_sV_{t+1}} \begin{pmatrix} V_1 & -V_0 \\ V_2 & -V_1 \end{pmatrix} \begin{pmatrix} V_t & -V_s \\ V_{t+1} & -V_{s+1} \end{pmatrix} \begin{pmatrix} V_{n+s} \\ V_{n+t} \end{pmatrix}$$

Theorem 16. (The Ultimate Identity) The fundamental identity connecting V_n and V_{n+1}

$$pV_nV_{n+1} + qV_n^2 - V_{n+1}^{\ 2} = (-q)^n \left(qV_1^2 + pV_1V_2 - V_2^2\right)$$

D'Ocagme' s Identity Identity Simson's.

Theorem 17. (Equivalent to Simson's Theorem)

For $V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}$, $n \geq 3$, $V_1 = a$, $V_2 = b$, with any coefficients *p* and *q*. The Ultimate identity connecting V_n and V_{n+m} is

$$K_mV_nV_{n+m} + (-1)^{m-1} q^mV_n^{\ 2} - V_{n+m}^{\ \ 2} = -q^nT_m^2\left(qV_1^2 + pV_1V_2 - V_2^2\right),$$

where

$$K_{n+2} = pK_{n+1} + qK_n, n \geq 0, K_0 = 2, K_1 = p \text{ and } \begin{pmatrix} T_m \\ T_{m-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{m-2} \begin{pmatrix} p \\ 1 \end{pmatrix},$$

and *m* is a constant.

Theorem 18. (Equivalent to Simson's Theorem) For any generalised second-order linear recurrence relation, $V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}$, $n \geq 3$, $V_1 = a$, $V_2 = b$, with any coefficients *p* and *q*. The Ultimate identity connecting V_n and V_{n+m} is

$$K_{n-m}V_mV_n + (-1)^{m-1} q^{n-m}V_m^{\ 2} - V_n^{\ 2} = -q^mT_{n-m}^2\left(qV_1^2 + pV_1V_2 - V_2^2\right)$$

$$K_{n+2} = pK_{n+1} + qK_n, n \geq 0, K_0 = 2, K_1 = p, \quad \begin{pmatrix} T_{n-m} \\ T_{n-m-1} \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^{n-m-2} \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

where *m* is a constant.

Theorem 19. (The General Recurrence) For any generalised second-order linear recurrence,

$V_n(p,q,a,b) = pV_{n-1} + qV_{n-2}$, $n \geq 3$, $V_1 = a$, $V_2 = b$, for any coefficients *p* and *q*,

$$V_{n+6} = \frac{\begin{array}{c}\left(V_{n+1}^2 - V_nV_{n+2}\right)V_{n+5}^2 + 2\left[\left(V_{n+3}V_n - V_{n+2}V_{n+1}\right)V_{n+4} - V_{n+1}V_{n+3}^2 + V_{n+3}V_{n+2}^2\right]V_{n+5} \\ -V_nV_{n+4}^3 + V_{n+4}^2V_{n+2}^2 - 3V_{n+4}V_{n+2}V_{n+3}^2 + V_{n+3}^4 + 2V_{n+4}^2V_{n+3}V_{n+1}\end{array}}{\left(V_{n+1}^{\ \ 2} - V_nV_{n+2}\right)V_{n+4} + V_nV_{n+3}^{\ \ 2} - 2V_{n+3}V_{n+2}V_{n+1} + V_{n+2}^{\ \ 2}}$$

$$\frac{V_{n+2}^{\ \ 2} - 2V_{n+1}V_{n+2}V_{n+3} + V_nV_{n+3}^{\ \ 2}}{V_nV_{n+2} - V_{n+1}^{\ \ 2}} = V_{n+4}$$

We call this the "universal recurrence" since it is satisfied by any generalised second-order linear recurrence with any coefficients *p* and *q*.

Conclusions

This study augments the continuing mathematical investigations of the generalised second-order linear recurrence relations. Generalised Fibonacci-like sequences are described in two distinct ways, originating new possibilities for further research into identities and applications. The study of Fibonacci numbers, originally introduced to Western European mathematics by Leonardo Pisano Bogollo, also known as Fibonacci, in his 1202 book *Liber Abaci*, has evolved into a rich area of mathematical investigation. The terms of any recognised sequence generated by second-order recurrence relations can be verified by imposing restrictions on the parameters p, q, a, and b. This work builds upon the introductory contributions of researchers given in the references, who explored various aspects of these sequences. Explicitly, this study introduces and analyses a generalised Fibonacci sequence, deriving its generating function, Binet's formula, and Simson's addition and subtraction formulas, as well as other formulas and identities, including those of Catalan and D'Ocagne. The derived formulas are applicable to several well-known sequences as special cases, including the classical Fibonacci, Lucas, k-Fibonacci, Pell, modified Pell, and Jacobsthal sequences, thus demonstrating the unifying power of the generalised approach. Various new results are obtained and studied within this generalised framework.

Conflict of Interest Statement

The authors declare that they have no conflict of interest.

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