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INNOVATIVE MATHEMATICAL MODELLING IN PREDATOR-PREY DYNAMICS: A SYSTEMATIC REVIEW

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ABSTRACT

Advancements in mathematical ecology have greatly expanded predator-prey models beyond the classical Lotka-Volterra framework, incorporating complex factors such as role reversal, fractional calculus, and the Allee effect. This review highlights innovative models and advanced mathematical techniques that enhance our understanding of predator-prey dynamics and their applications across disciplines. It analysed recent studies introducing new ecological factors, including maturation delays, disease dynamics, and spatial heterogeneity. Techniques like fractional calculus and network theory were examined for their effectiveness in capturing complex behaviours. Models addressing role reversal between adult prey and juvenile predators or incorporating generalized fractional derivatives reveal significant impacts on population stability. Additionally, maturation delays, handling time, and gestation periods markedly influence oscillatory dynamics. The reviewed models demonstrate versatility in guiding pest control, understanding disease spread, and optimizing biotechnological processes. This review shows that modern predator-prey models, enriched by complex ecological factors and advanced mathematics, provide profound insights into system dynamics, with practical applications across various fields. As global challenges grow, these models offer crucial guidance for developing more resilient and sustainable systems, underscoring their potential to address issues like food security, disease outbreaks, and ecosystem degradation. Future research should integrate emerging factors, such as anthropogenic noise and industrial pollutants, to further enhance the models' real-world relevance.

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Introduction

Mathematical modelling has become an indispensable tool in understanding the dynamics of predator-prey interactions. These models provide insights into complex ecological relationships and help in developing effective management strategies. This review synthesises recent advancements in predator-prey modelling, highlighting innovative approaches and their contributions to the field of mathematical ecology.

One novel approach incorporates role reversal, where adult prey attack and kill juvenile predators. Derived from the McKendrick–Von Foerster equation, this model includes a maturation delay for juvenile predators, which significantly impacts population dynamics by preventing cyclic behaviours [1]. Comparative analysis with a model lacking role reversal shows that maturation delay and handling time significantly impact the dynamics, suggesting that role reversal could prevent cyclic population dynamics. The theoretical predictions align with available data on forage-piscivore fish trade-offs, although experimental validation is necessary [1].

A different perspective is offered by a fractional prey-predator model using the generalised Hattaf fractional derivative (GHF), demonstrating ecological and mathematical well-posedness with three equilibrium points whose stability is established [2]. This approach enriches the understanding of predator-prey dynamics through fractional calculus, providing a robust framework for analysing ecological systems.

The introduction of a three-dimensional prey-predator model including disease in predators and a gestation time delay adds another layer of complexity. The model's positive invariance and the existence of equilibria are established, with local stability analysed for both delayed and non-delayed systems. The investigation into Hopf bifurcation and periodic solutions underscores the critical role of time delay in controlling system dynamics, validated through numerical simulations [3].

Exploring eco-epidemiological dynamics, a modified Lotka-Volterra model incorporates SI epidemic dynamics within prey-predator interactions. Stability analysis using Lyapunov functions and next-generation matrix calculations reveals the conditions for stable disease-free equilibrium. Numerical simulations confirm these stability results, highlighting the effects of infection and attack rates on system behaviour [4].

The extension of the Lotka-Volterra framework to a four-species predator-prey model examines the interactions of multiple prey and predator species. Using Takagi-Sugeno (T-S) and fuzzy impulsive control models, the global stability and dynamics of the system are explored. These methods effectively address non-linearities and uncertainties in complex ecological systems [5].

In addressing specific ecological contexts, a model for brown planthopper (BPH) infestation in rice under varying habitat complexity and monsoon conditions employs first-order differential equations and aggregated methods. The model's positivity and boundedness are demonstrated with equilibrium points and local stability were analysed. Hopf bifurcation was investigated, and numerical simulations illustrate theoretical findings, contributing to ecological management strategies [6].

Further, a two-intra-species predator-prey model based on the Lotka-Volterra framework utilizes T-S and fuzzy impulsive control models to investigate stability. Global stabilities and fuzzy solutions are derived through numerical simulations, offering insights into the intricate dynamics of predator-prey systems [7].

Incorporating the Allee effect into a predator-prey model reveals its significant impact on population growth. The study explores stability and bifurcation properties, showing that slight parameter variations can lead to chaotic behaviour, validated through numerical simulations. This highlights the importance of the Allee effect in ecological modelling [8].

Prey taxis models with prey-induced acceleration provide another innovative approach. Unlike conventional prey taxis systems, these models consider predator acceleration proportional to prey density gradient, better explaining spatial heterogeneity. Stability and patterning regimes are analysed through linear stability and numerical simulations, demonstrating the model's efficacy in capturing complex predator-prey interactions [9].

Lastly, a four-dimensional deterministic model of human immune response to viral infections uses first-order differential equations to analyse local behaviour through dynamical systems theory. Introducing control patterns yields significant theoretical and medical insights, emphasising the model's relevance to real-world biological systems [10]. These diverse approaches and models underscore the richness and complexity of predator-prey dynamics, each contributing unique insights and methodologies to the field of mathematical ecology.

The comparison Table 1 provides a detailed overview of several key predator-prey models used in mathematical ecology. Each model is described in terms of its unique properties and how it differs from other models. This table is designed to help researchers quickly identify the strengths and limitations of each model, facilitating the selection of the most appropriate model for their specific research needs. By highlighting the distinctive features and applications of each model, the table aims to enhance the understanding of how different mathematical approaches can be used to study predator-prey dynamics.

Table 1: Comparison of various predator-prey models

Model	Properties	Differences
McKendrick-Von Foerster	Age-structured population dynamics.	Incorporates age structure, unlike other models.
Fractional Model	Captures systems with memory and hereditary properties.	More accurate for systems with memory effects compared to integer-order models.
Lotka-Volterra	Describes cyclical predator-prey population dynamics.	Simplest and assumes constant interaction rates.
Takagi-Sugeno and Fuzzy Impulsive Control	Handles uncertainties and sudden changes.	Manages non-linearities and uncertainties better than deterministic models.
Predator-Prey with Allee Effect	Population growth rate increases with density at low densities.	Suitable for populations with low density struggles.
Prey Taxis with Prey-Induced Acceleration	Focuses on spatial predator movement.	Captures spatial heterogeneity unlike homogenous models.
Deterministic Model	Assumes fixed laws without random fluctuations.	Lacks the ability to incorporate randomness.
Holling Types I and II	Describe functional response of predators to prey density.	Type II more realistic, considering handling time.

Literature Review

The field of mathematical ecology has seen significant advancements in modelling predator-prey interactions, with researchers exploring various factors that influence these complex dynamics. This review synthesizes recent studies, focusing on key themes that shape our understanding of these intricate ecological relationships.

Ecological Relationships

A central aspect of predator-prey models is the choice of functional response, which describes how predation rate changes with prey density. Alebraheem and Abu-Hassan [11] clarify the relationship between models and Holling types functional and numerical responses, providing insights into predator interferences and competition mechanisms. Similarly, Tiwari *et al*. [12] incorporate Cosner-type functional response to model schooling or conspecific aggregation behaviour of predators, emphasizing strong Allee effects. Kumar and Gunasundari [13,14] explore models with Holling type I and II functional responses, respectively, demonstrating their impact on system stability and dynamics.

The Allee effect, where population growth rate is positively related to population size at low densities, significantly influences predator-prey dynamics. Tiwari *et al*. [12] show that under strong Allee effects, predator-prey populations form high-density schools, a finding consistent with biological observations. Umrao and Srivastava [15] investigate a model with fear and Allee effects in prey, revealing that the system can experience persistence, bi-stability, or tri-stability depending on the Allee effect's strength. Kumar and Gunasundari [14] further demonstrate how an additive Allee effect in the predator population affects system stability and equilibrium points.

Ecological models increasingly consider the impact of disease and invasive species on predator-prey dynamics. Thota and Ayoade [16] propose a prey-diseased predator model with refuge in prey, analysing disease-free and endemic equilibria. Das *et al*. [17] study the dual impact of harvesting and invasive species, concluding that invasions always harm the ecosystem's biological conservation. Their results suggest that predator-oriented harvesting provides optimal yield and resilience in invaded systems.

Harvesting significantly affects predator-prey dynamics, with implications for conservation. Elmojtaba *et al*. [18] show that high harvesting rates can lead to extinction, but optimal harvesting can provide high yields while preserving populations. Das *et al*. [17] find that in systems with invasive species, predator-oriented harvesting with low predator catchability offers the best balance between yield and resilience.

Mathematical Techniques and Numerical Methods

Recent studies employ a variety of sophisticated mathematical techniques. This section aims to explore the various mathematical techniques and numerical methods used in predator-prey modelling, highlighting their advantages and specific applications. Mathematical techniques and numerical methods are crucial in solving complex ecological models that cannot be addressed analytically. These methods include fractional calculus, numerical simulations, and advanced computational algorithms.

Ali *et al*. [19] use the Caputo-Fabrizio fractal-fractional order operator, revealing that small immigrations can stabilize predator-prey ecosystems. This technique allows for modelling systems with memory effects, providing a more accurate description of real-world dynamics. Alzaid *et al*. [20] and Ramesh *et al*. [21] apply fractional calculus to enrich model dynamics. Fractional calculus is particularly useful for capturing long-term dependencies and hereditary properties of ecological systems.

Ramesh *et al*. [21] demonstrate's that delayed fractional order with harvesting can adjust species biomass effectively, offering new insights into resource management and conservation. Mungkasi [22] proposes a new analytical-numerical method that outperforms existing techniques in speed and accuracy. This method integrates analytical solutions with numerical algorithms, enhancing computational efficiency and precision. The hybrid approach leverages the strengths of both analytical and numerical methods, reducing computational time while maintaining high accuracy. This makes it particularly suitable for large-scale ecological models where traditional methods may fall short.

Stability Analysis and Bifurcations

This section discusses the importance of performing stability analysis and bifurcation studies in ecological modelling and clarifies the differences between these two approaches. Stability analysis and bifurcation theory are essential for understanding the behaviour of ecological models under various conditions. Stability analysis determines whether a system returns to equilibrium after a disturbance. It helps in identifying stable and unstable equilibrium points, providing insights into the resilience of ecological systems.

Bifurcation analysis studies how the qualitative structure of a system changes as parameters vary. This is crucial for understanding how small changees in environmental or biological parameters can lead to significant changes in system behaviour, such as the transition from stable to chaotic dynamics. Stability analysis focuses on the local behaviour of the system around equilibrium points, determining if perturbations decay or grow over time, while bifurcation studies examine the global behaviour of the system, identifying parameter values where qualitative changes occur, such as the appearance of periodic solutions or chaos.

Vijayalakshmi and Senthamarai [23] use Lyapunov functionals to analyse stability in a system with imprecise parameters, providing a robust framework for ensuring model reliability under uncertainty. Pavan Kumar *et al*. [24] investigate stochastic stability and saddle-node bifurcation in a model where prey has nonlinear reproduction, highlighting how environmental variability affects population dynamics. Tiwari *et al*. [12] and Umrao and Srivastava [15] observe various bifurcations, including saddle-node, Hopf, and Bogdanov-Takens, demonstrating the complex dynamics that can arise from parameter changes in predator-prey models.

Ecosystem-Specific Models

This section aims to clarify the classification of ecosystem-specific models and provide guidelines for such classification. Ecosystem-specific models are tailored to particular ecological contexts, taking into account unique interactions and environmental conditions. Models are classified based on the types of species interactions they represent, such as predator-prey, mutualism, or competition; consideration of specific environmental factors, such as habitat complexity, climate, and resource availability; and the focus on particular ecological processes, such as nutrient cycling, energy flow, or population dynamics, that are unique to the ecosystem in question.

Khansai *et al*. [6] model studies brown plant hopper infestation in rice, considering habitat complexity and monsoon conditions, which are critical for understanding pest dynamics in agricultural systems. Middlebrook and Wang [25] study looked at stoichiometric models of salmon, bears, and vegetation, highlighting how different growth rates and nutrient availability affect species coexistence in riparian ecosystems. Suganya and Senthamarai [26] investigated phytoplankton-zooplankton-nanoparticle interactions with density-dependent predator death rates, providing insights into aquatic ecosystem dynamics.

Yadav and Kumar [27,28] developed a model for banana-nematodes dynamics to assist farmers in pest control, focusing on specific agricultural practices and pest management strategies. The reference to Yadav and Kumar [27,28] focuses on the application of fuzzy logic in ecological modelling, specifically addressing pest control strategies in banana plantations.

Methodology

Predator-prey modelling is currently the subject of several studies conducted all over the world. This section discusses the necessity for a systemic understanding of how new mathematical tools, interdisciplinary applications, and predator-prey models are developing mathematical ecology. By contrast, the approach to answering the research questions that the present study posed was described in the section before this one. This review is divided into four parts:

I. Innovative Predator-Prey Models

II. Cross-Disciplinary Applications

III. Advanced Mathematical Techniques

IV. Novel Perspectives and System-Level Insights

The scientific literature is then methodically reviewed and synthesised in this section to identify, pick, and evaluate how these themes affect our comprehension of predator-prey dynamics and their wider ramifications. Finally, the discussion considers future research directions and practical applications to address what should be done in response to the issues raised. In this analysis, the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) approach a published guideline for carrying out a systematic literature review is utilized. Publication requirements are typically required to help writers evaluate and examine the rigor and accuracy of a review with pertinent and essential facts. The randomized studies evaluations survey is another important tool that PRISMA emphasises and can play a significant role in systematic analysis reports for various research formats [29]. (Picture 1). Given their robustness, two important databases i.e. Scopus and Web of Science (WOS)—were used as instruments to assess the research's approach. Numerous investigations were discussed, including ones involving mathematical modelling. But no database is flawless and comprehensive, just like WOS and Scopus [30]. A summary of the four important sub-sections; identification, screening, eligibility, and data abstraction is also given in this section.

Identification

The systematic review process consists of three basic phases that were used to choose many relevant papers for this study. The first phase entails the identification of keywords and the search for associated related terms using thesaurus dictionaries, encyclopaedias, and prior research. Following the selection of all pertinent terms, search strings for the databases Scopus and Web of Science

(Table 2) have been made. The current research endeavour effectively retrieved 646 papers from both databases during the first stage of the systematic review process.

Table 2: The search strings

Scopus	TITLE-ABS-KEY ((mathematical modelling OR “mathematical model” OR “mathematical models”) AND (“prey and predator model” OR “prey-predator model” OR “predator-prey model”)) AND PUBYEAR > 2019 AND PUBYEAR < 2025 AND (LIMIT-TO (EXACTKEYWORD , “mathematical models”) OR LIMIT-TO (EXACTKEYWORD , “mathematical model”) OR LIMIT-TO (EXACTKEYWORD , “predator-prey model”) OR LIMIT-TO (EXACTKEYWORD , “predator prey systems”) OR LIMIT-TO (EXACTKEYWORD , “predator-prey modelling”) OR LIMIT-TO (EXACTKEYWORD , “bifurcation (mathematics)”) OR LIMIT-TO (EXACTKEYWORD , “bifurcation”) OR LIMIT-TO (EXACTKEYWORD , “prey-predator models”) OR LIMIT-TO (EXACTKEYWORD , “prey-predator model”)) AND (LIMIT-TO (SUBJAREA , “math”)) AND (LIMIT-TO (DOCTYPE , “ar”)) AND (LIMIT-TO (LANGUAGE , “English”)) Access Date: 10 May 2024
WOS	(“Lotka-Volterra model” OR “prey-predator model” OR “predator-prey model”) AND (“mathematical model*” OR “mathematical modelling”) (All Fields) and 2024 or 2023 or 2022 or 2021 or 2020 or 2019 or 2018 (Publication Years) and Review Article (Exclude – Document Types) and Article (Document Types) and English (Languages) Access Date: 16 May 2024

Screening

Duplicate papers were disqualified during the initial screening. 563 papers were rejected in the first stage, and 68 papers were reviewed in the second stage based on a diverse inclusion and exclusion criteria. Literature (research articles) was the first criterion used because it is the primary source of useful guidance. Aside from the most recent research, it also includes systematic reviews, reviews, meta-synthesis, meta-analysis, books, book series, chapters, and conference proceedings. In addition, the review was restricted to works published in English. Remember that the strategy was created for the recent year (2020-2024). 197 publications in all were disqualified based on specific standards.

During the initial screening process, duplicated papers were deliberately excluded. This ensured that only unique and distinct articles were considered for further analysis. In this first phase, a total of 10 articles were omitted due to their duplication. Moving on to the second phase, a thorough screening was conducted on 83 articles. This screening process involved the application of various inclusion and exclusion criteria that had been carefully developed by the researchers. One of the primary criteria used was the consideration of literature, specifically research articles, as the main source of practical information. Consequently, any publications in the form of systematic reviews, reviews, meta-analyses, meta-syntheses, book series, books, chapters, and conference proceedings were excluded from the current study. Furthermore, the review focused exclusively on papers written in English, as it is a widely accepted language in academic circles. This decision was made to ensure consistency and coherence in the analysis process. It is important to note that the chosen timeframe for this study was a duration from 2020 to 2024. This timeframe allowed for a comprehensive examination of the relevant literature within a specific period. In total, 197 publications were excluded based on the specific parameters set forth in the screening process. These rigorous criteria were implemented to ensure the selection of high-quality and relevant articles for the subsequent stages of the research.

Eligibility

During the third stage, which is known as the eligibility assessment, 83 articles were compiled. At this point, in order to verify that the publications matched the inclusion criteria and were pertinent to the ongoing study goals, we carefully reviewed the titles and main body of each one. As a result, 46 papers were deemed irrelevant to the research domain, had names that were not noteworthy, or had summaries that were pertinent to the goals of the study as supported by empirical data. Consequently, 50 articles were kept for additional review (see Table 3 for specifics).

Table 3: The selection criterion is searching

Criterion	Inclusion	Exclusion
Language	English	Non-English
Timeline	2020 – 2024	< 2022
Literature type	Journal (Article)	Conference, Book, Review
Publication Stage	Final	In Press

Data Abstraction and Analysis

This study examined and synthesized a range of research designs (quantitative, qualitative, and mixed techniques) using an integrative analysis as one of the assessment strategies. Finding pertinent subjects and subtopics was the aim of the competent study. The process of gathering data marked the beginning of the theme's growth. Figure 1 illustrates the method by which the authors carefully examined a collection of fifty publications for claims or information pertinent to the subjects of the present investigation. The important recent research on the application of e-learning was then assessed by the writers. Both the research findings and the methods applied in each study are under investigation.

Next, the author collaborated with other co-authors to develop themes based on the evidence in this study's context. A log was kept throughout the data analysis process to record any analyses, viewpoints, riddles, or other thoughts relevant to the data interpretation. Finally, the authors compared the results to see if there were any inconsistencies in the theme design process. It is worth noting that if there are any disagreements between the concepts, the authors discuss them among themselves. The produced themes were eventually tweaked to ensure consistency. The analysis selection was carried out by two experts, one in public health (Khairul Shakir Ab Rahman expert medical doctor in pathology) and the other in biomedical science (Wan Azani Mustafa expert in biomedical computing) to determine the validity of the problems. The expert review phase ensures the clarity, importance, and suitability of each subtheme by establishing the domain.

Results and Discussions

The findings demonstrate the versatility, depth, and real-world applicability of these models. Four dominant sections emerge from the analysis: Innovative Predator-Prey Models, Cross-Disciplinary Applications, Advanced Mathematical Techniques and Novel Perspectives and System-Level Insights.

Innovative Predator-Prey Models

Innovative predator-prey models are those that incorporate new mechanisms, interactions, or complexities beyond traditional models. These innovations can include non-linear dynamics, role reversals, adaptive behaviours, or multi-species interactions that provide a deeper understanding of ecological relationships and better reflect real-world scenarios [31-33]. Models are categorised as innovative predator-prey models based on their incorporation of non-linear dynamics, which include non-linear responses, feedback loops, or other complexities that traditional linear models do not capture [8,9]. Additionally, models that consider scenarios where traditional roles of predator and prey are reversed or where species exhibit adaptive behaviours in response to environmental changes are also classified as innovative [11]. Multi-species interactions extend beyond simple two-species interactions to include multiple predators, prey, or other interacting species, thus capturing the complexity of real ecosystems [7,34]. Furthermore, models that apply sophisticated mathematical or computational techniques, such as fractional calculus, network theory, or agent-based modelling, to analyse and predict ecological dynamics, are considered innovative [35].

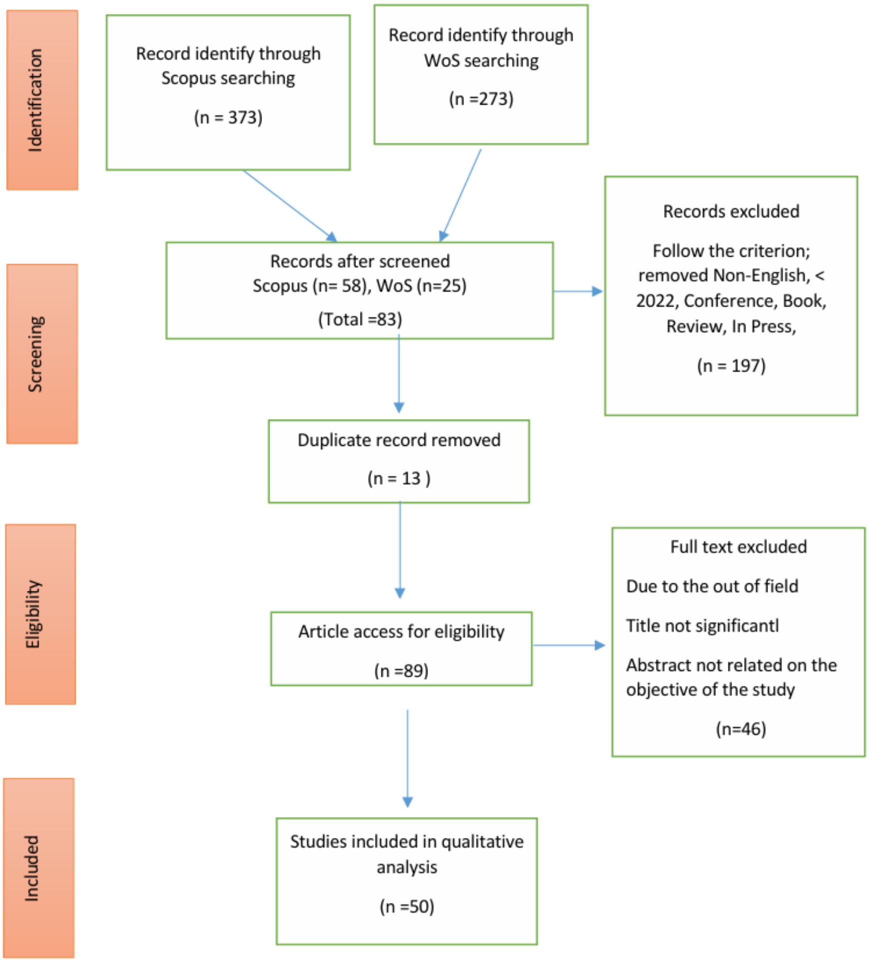


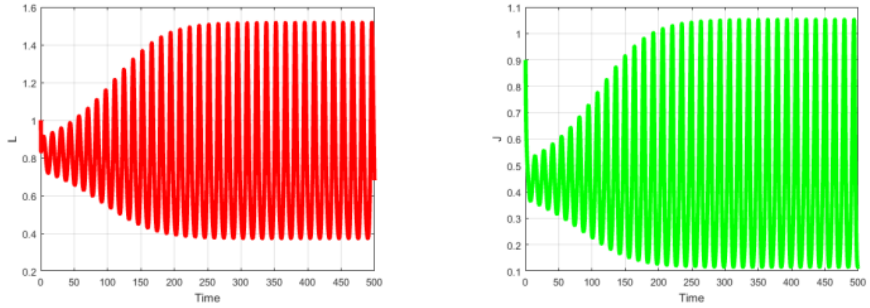
Figure 1: Flow diagram of the proposed searching study [29]

Recent research has significantly expanded the classical Lotka-Volterra framework, incorporating complex ecological factors. Misra and Yadav [31] introduced a model where insect population growth depends entirely on crops, using Holling type-II response. Their results show that while crop consumption rate destabilizes the system, insecticide spraying rate stabilizes it. Similarly, Trujillo-Salazar *et al*. [32] modelled coffee berry borer control through ant predation, predicting different dynamics as parameters vary.

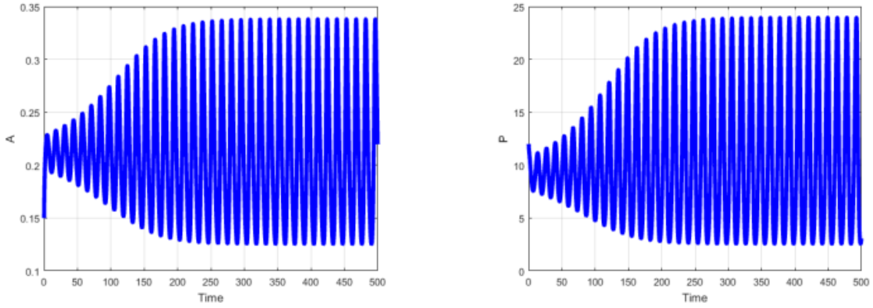
Han *et al*. [33] modified the classic model by describing the rate of change for each species, showing that coupling terms greatly expand the dynamics, making it more applicable to real-world phenomena. Bolohan *et al*. [35] broke new ground by modelling seasonal dynamics between a generalist and a specialist predator on a single prey. Their work is crucial as global change alters season lengths, affecting community dynamics.

The introduction of additional complexities like the Allee effect by Akhtar *et al*. [8] reveals that slight parameter variations can lead to chaotic behaviour. Similarly, Mu *et al*. [9] showed that prey-induced predator acceleration better explains spatial heterogeneity. These findings highlight how even small changes in model parameters can profoundly affect system dynamics.

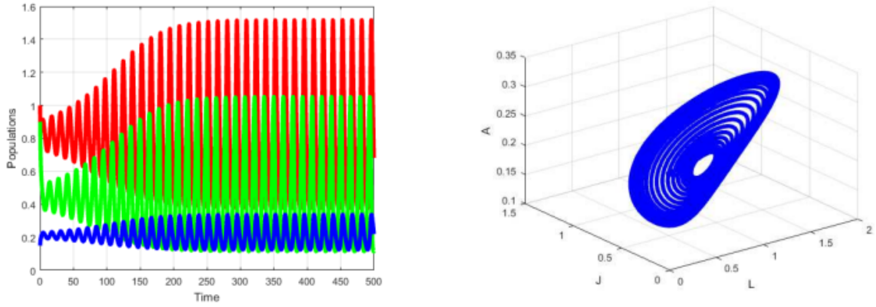
Further innovations include Alebraheem and Abu-Hassan [11] model, which assumes the predator's carrying capacity depends on prey availability, revealing how agricultural crop consumption rate and insecticide spraying rate affect stability. Kaladhar and Singh [7] examine two-predator, one-prey dynamics, focusing on competition and Allee effects, while Ouedraogo *et al*. [34] model fish and plankton dynamics through a trophic chain predation, offering insights into different fishing areas. Figure 2 in their paper is a striking visual representation of this trophic chain predation model. The phase portrait of the Larva-Juvenile-Adult system shows how larvae feed on plankton, juveniles prey on larvae, and adults consume juveniles. This cascading predation structure creates a complex web of interactions, influencing the population dynamics at each stage. The spiralling trajectories in the phase space reveal stable limit cycles, indicating that despite this predatory hierarchy, the three fish populations can coexist in a balanced, oscillatory state.



Larva trajectories. Juvenile trajectories.



Adult trajectories. Plankton trajectories.



Larva-Juvenile-Adult model trajectories. Phase portrait of Larva-Juvenile-Adult system.

Figure 2: Fish population dynamics under moderate fishing (β=0.205), showing stable oscillations (Ouedraogo *et al*., 2019).

Cross-Disciplinary Applications

Cross-disciplinary applications of predator-prey models extend their utility beyond traditional ecological studies, integrating concepts and techniques from various fields such as epidemiology, environmental science, economics, and resource management. By applying these models in diverse contexts, researchers can gain insights into complex systems and address broad societal challenges. This section reviews key studies that exemplify the application of predator-prey models across different disciplines.

In this context, MD refers to the application of predator-prey models across multiple disciplines to solve complex problems that involve interacting systems. These models help bridge gaps between different scientific fields, providing a unified framework to understand and manage dynamic interactions in various contexts.

The power of predator-prey models extends far beyond ecology. In agriculture, Yadav and Kumar [27,28] applied these models to control nematodes in banana crops and increase sugarcane and vegetable production. Their work demonstrates that optimal control strategies, like releasing infected predators, can effectively manage pests.

In epidemiology, Cheru *et al*. [4] modelled infectious disease in prey-predator systems, showing that domestic and industrial pollution sources affect disease dynamics. Melese and Kiros [36] went further, modelling migration and vaccination in a system where both prey and predator are affected by disease. Their findings suggest that vaccination and migration rates significantly impact stability.

Even in biotechnology, Shi *et al*. [37] used a Lotka-Volterra model to enhance vitamin B_{12} content in fermented soymilk. They found that when the interaction coefficients between two microorganisms' approach one, vitamin B_{12} production is maximized. In oncology, Sultana *et al*. [38] compared predator-prey and game theory models to understand leukaemia dynamics, using these insights to discuss stem cell transplant strategies.

Plant-insect interactions are crucial for agriculture, are also modelled. Kumar *et al*. [39] took a molecular approach, using computational modelling and MD simulations to understand these interactions at the protein level. Their model predicts how BIK1 inhibition can improve plant quality, showcasing the model's potential in genetic engineering.

Advanced Mathematical Techniques

Advanced mathematical techniques in predator-prey modelling are those that incorporate sophisticated methodologies, which go beyond traditional linear and non-linear models. These techniques often involve higher-order mathematical concepts, computational algorithms, and complex system analyses that provide deeper insights and more accurate predictions. Techniques are categorized as advanced based on their complexity, the level of mathematical abstraction, and their ability to address intricate ecological dynamics that simpler models cannot capture.

Properties of Advanced Mathematical Techniques

Fractional calculus extends the concept of integer-order derivatives and integrals to non-integer (fractional) orders, allowing for the modelling of systems with memory effects and hereditary properties. This technique provides a more accurate representation of processes where current states depend on historical data. Network theory involves the use of graph-based models to represent and analyse the interconnections between different components of an ecosystem. This technique is particularly useful for studying complex interactions and dependencies within ecological networks, such as food webs and mutualistic relationships. Agent-based modelling is a computational technique that simulates the actions and interactions of individual agents (e.g., organisms) to assess their effects on the system as a whole. This method is effective for capturing emergent behaviours and adaptive strategies in heterogeneous populations.

Panja [40] found that fractional-order derivatives can improve system stability by providing a more accurate representation of the system's memory and hereditary properties. Theories behind these findings suggest that fractional calculus captures the long-term dependencies and intrinsic complexities of ecological interactions, leading to enhanced stability and predictability. Specifically, fractional-order models can account for the influence of past states on current dynamics, which is crucial for systems where the effects of previous interactions persist over time. This approach allows for a more nuanced understanding of stability, as it incorporates the continuous influence of historical events on the system's behaviour.

Researchers are increasingly employing sophisticated mathematical tools to capture complex dynamics. Fractional calculus is gaining prominence; Assadiki *et al*. [2] and Ramesh *et al*. [21] showed that fractional-order derivatives enrich model dynamics and can adjust species biomass effectively. Panja [40] found that fractional-order derivatives can improve system stability in a three-species model.

Network theory is another emerging field. Guo *et al*. [41] studied a one-predator, two-prey model on an Erdős-Rényi network, showing that as nodes and edges increase, the system tends toward chaos earlier. This insight is crucial for understanding how network complexity affects ecological stability.

Stability analysis techniques are becoming more sophisticated. Kharbanda and Kumar [42] used Lyapunov functions to prove global asymptotic stability in a three-species model. Vijayalakshmi and Senthamarai [23] combined Lyapunov functionals with homotropy perturbation and variational iteration methods for a nonlinear imprecise model. Impressively, Mungkasi [22] developed a new analytical-numerical method that outperforms existing techniques in speed and accuracy.

Time delays are increasingly recognized as critical. El Foutayenil and Khaladi [43] showed how multiple delays affect fish population dynamics, while Toro-Zapata *et al*. [44] used delay terms to model CD4 T-cell activation in HIV infection. In ecology, Al-Jubouri and Naji [45] incorporated delays in disease transmission from diseased predators to healthy ones, revealing complex dynamics.

The use of fuzzy logic and impulsive control is another frontier. Singh and Kolla [5] use these techniques in a four-species model, showing that more complex networks exhibit chaotic behaviour earlier. This approach is particularly effective in handling nonlinearities and uncertainties in ecological systems.

Novel Perspectives and System-Level Insights

System-level analysis refers to the comprehensive examination of an entire ecological system, considering the interactions and relationships between all its components. This approach goes beyond analysing individual species or isolated interactions, instead focusing on how different parts of the system work together to influence overall dynamics and outcomes. System-level insights are crucial for understanding the broader implications of ecological interactions and for developing strategies that account for the complexity and interconnectedness of ecosystems [46].

The integration of advanced mathematical techniques like fractional calculus, network theory, and improved stability analysis methods is pushing the boundaries of what these models can achieve. This means that by incorporating these sophisticated methods, researchers can explore more complex and realistic scenarios within ecological models. Fractional calculus, for example, allows for the modelling of systems with memory effects and hereditary properties, providing a more accurate representation of processes where current states depend on historical data [2,21]. Network theory offers powerful tools for analysing the interconnections within ecosystems, such as food webs and mutualistic relationships, helping to understand how changes in one part of the system can ripple through the entire network [41]. Improved stability analysis methods, including Lyapunov functions and bifurcation analysis, provide deeper insights into the conditions under which ecological systems remain stable or transition to new states [23,42]. Together, these techniques enable the creation of models that are not only more accurate but also capable of addressing the complex, non-linear, and dynamic nature of real-world ecosystems. This integrated approach enhances our ability to predict, manage, and conserve ecological systems in the face of changing environmental conditions [5,43].

Several studies offer unique perspectives that challenge conventional thinking. Shi *et al*. [37] model phytoplankton-zooplankton-nanoparticle interactions, highlighting how industrial pollution affects ecosystem dynamics. Their work underscores the need to address both domestic and industrial wastewater treatment.

Ramirez-Carrasco *et al*. [47] introduce noise into metapopulation models, reflecting the impact of anthropogenic sound. Their findings suggest that as noise intensity increases, differences in carrying capacities produce noticeable, long-term differences between subpopulation sizes—a critical insight as human activities increasingly alter soundscapes.

Broader system-level insights are also emerging. Panayotova *et al*. [48] used predator-prey models to study the Chesapeake Bay fisheries, revealing that delayed fractional order where harvesting can effectively adjust species biomass. Suganya and Senthamarai [26] extend this to phytoplankton-zooplankton-nanoparticle systems, showing how nanoparticles influence spatial dynamics.

In summary, this review reveals a field that is not only theoretically rich but can also increasingly applied across disciplines. From ecology to epidemiology, agriculture to oncology, predator-prey models are providing critical insights. The integration of advanced mathematical techniques like fractional calculus, network theory, and improved stability analysis methods is pushing the boundaries of what these models can achieve. As global challenges like climate change, disease outbreaks, and food security intensify, the findings from these models offer a beacon of hope, guiding us toward more resilient and sustainable systems.

Conclusions

In conclusion, advanced mathematical models in predator-prey dynamics offer profound insights into ecological, agricultural, and epidemiological systems. Incorporating role reversal, fractional derivatives, and disease dynamics enrich classical frameworks, providing deeper understanding and management strategies. Cross-disciplinary applications extend to agriculture, epidemiology, and biotechnology, offering solutions for pest control, disease management, and production enhancement. Advanced techniques like fractional calculus and network theory push the boundaries of ecological modelling, capturing complex dynamics. As global challenges escalate, these models provide essential guidance for resilience and sustainability. Continued development and application of these models are critical for addressing contemporary ecological and societal challenges.

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Conflict of Interest Statement

The authors declare that they have no conflicts of interest to report regarding the present study.

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