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DYNAMICAL BEHAVIOUR OF A DISCRETE-TIME PREDATOR-PREY MODEL WITH INTRAGUILD PREDATION

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ABSTRACT

This article examines a discrete-time predator-prey model that incorporates intraguild predation. For biological reasons, positivity and boundedness of solutions are verified. A condition for the stability of an interior fixed point is derived. Global stability criterion of the interior fixed point is obtained. It is found that the system exhibits Neimark-Sacker and period-doubling bifurcations under certain restrictions on the system parameters. The system shows a chaotic nature for a particular choice of system parameters. This phenomenon may be prevented by applying a hybrid control technique. We have observed that an increase in the amount of intrinsic growth rate of prey initially destabilises the system and finally stabilises, whereas an increase in the amount of intraspecific competition rate of prey cannot stabilise the system. Some illustrations support our analytical findings.

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Introduction

The dynamic correlation between prey and predator living in the same environment is one of the most crucial and fascinating topics in the field of mathematical biology. Predator-prey models of various types are extensively studied in [1, 2, 3, 4, 5, 6, 7, 8, 9]. Although in a few circumstances, predator and prey depend on the same resource species. This relation may be referred to as intraguild predation (IGP) [10, 11, 12, 13]. It has a major impact on the management and conservation of species [14]. IGP systems are investigated through mathematical models by imposing restrictions on the interactions between nutrient levels, utilising predation processes such as Holling type I, II, and III [15, 16, 17, 18]. The impact of IGP on the aquatic environment is evident in [19, 20]. In [21], Safuan *et al*. analysed an IGP model by considering the density-dependent carrying capacity of both the prey and the predator. They observed coexistence, limit cycle, and extinction in the system under certain restrictions.

Discrete-time models can convey the true picture of the system, even when observation spacings are big. In comparison, continuous-time models are only correct when observation spacings are small. In the case of non-overlapping generations, it is more appropriate to consider a discrete version of a continuous system. In nature, fish and insect populations follow this type of generation. Furthermore, complex dynamics can be observed in discrete-time models rather than continuous models [22, 23, 24, 25]. Discrete-time predator-prey models can exhibit chaotic behaviour [22, 23]. Further fascinating and important results on discrete prey-predator models are available in [26, 27, 28, 29, 30].

Most of the earlier studies address two species. Complex dynamics emerge when multiple species coexist. It is very interesting to understand the complex dynamics of the discrete-time IGP model. So, we suggest a discrete-time predator-prey system with intraguild predation. We investigate the feasibility and local as well as global stability of the interior fixed point. Next, we determine the conditions on system parameters that establish Neimark-Sacker (NS) and period-doubling (PD) bifurcations. A measure for controlling chaos will be indicated. Sen *et al*. [31] explored the dynamics of the model given below:

$$\begin{aligned} \frac{dx}{dt} &= x(r_1 - a_{11}x - a_{12}y - a_{13}z), \\ \frac{dy}{dt} &= y\left(a_{21}x - a_{22}y - \frac{a_{23}z}{1+hy} - r_2\right), \\ \frac{dz}{dt} &= z\left(a_{31}x - \frac{a_{32}y}{1+hy} - a_{33}z - r_3\right). \end{aligned} \tag{1}$$

Here, x, y, and z denote the biomasses of prey, intermediate predator, and intraguild predator at time, respectively. The meaning of system parameters and symbols are given in Table 1. The parameters are taken to be positive.

Table 1: Parameter description

Parameters	**Biological Significance**
r_1	The rate of intrinsic growth of x
a_{11}	Intraspecific competition rate in x
a_{12}	The consumption rate of y
a_{13}	The consumption rate of z
r_2	The mortality rate of y
a_{21}	Prey consumption into reproduction for intermediate predator
a_{22}	Intraspecific competition rate in y
a_{23}	The rate of predation of z sustaining upon y
r_3	The natural mortality rate of z
a_{31}	Conversion efficiency of z from x
a_{32}	Conversion efficiency of z from y
a_{33}	Intraspecific competition rate in z
h	The reciprocal of half the half-saturation constant

Basic dynamics include global stability, bifurcation, and chaos control when system (1) is taken in discrete form. To analyse it, a discrete version of the continuous model (1) is given below:

$$\begin{aligned} x_{n+1} &= x_n \exp(r_1 - a_{11}x_n - a_{12}y_n - a_{13}z_n), \\ y_{n+1} &= y_n \exp\left(a_{21}x_n - a_{22}y_n - \frac{a_{23}z_n}{1+hy_n} - r_2\right), \\ z_{n+1} &= z_n \exp\left(a_{31}x_n - \frac{a_{32}y_n}{1+hy_n} - a_{33}z_n - r_3\right). \end{aligned} \tag{2}$$

where x_n, y_n, and z_n stands for the biomass of prey, intermediate predator, and intraguild predator at generation $n \in \mathbb{N}$.

The remainder of the article is organised in this manner. In Section 2, we show that the solutions are bounded and positive. The stability condition of the interior fixed point is described in Section 3. A global analysis of the interior fixed point is presented in Section 4. NS and PD bifurcation are discussed in Section 5. Section 6 deals with how chaos is regulated. Numerical simulations are conducted in Section 7. Concluding remarks are presented in Section 8.

Bounds and Positiveness of Solutions

In this section, we demonstrate the positivity and boundedness of solutions to system (2). The proof of the first proposition is trivial, and its evidence is not given.

Proposition 1. For any positive initial values, solutions of system (2) are positive.

We use the following proposition to establish whether the solutions of system (2) are bounded or not.

Proposition 2. [32] Assume that x_m fulfils $x_0 > 0$ and $x_{m+1} \leq x_m\, exp[p(1 - qx_m)]$ for $m \in [m_1, \infty)$ where $q > 0$. Then, $limsup_{n\to\infty}\, x_m \leq \frac{1}{pq} exp(p - 1)$.

Uniform boundedness for system (2) is confirmed in the Theorem 1 stated below.

Theorem 1. Every positive solution $\{(x_n, y_n, z_n)\}$ of system (2) is uniformly bounded.

Proof. For an arbitrary positive solution $\{(x_n, y_n, z_n)\}$ of system (2), from the beginning equation of system (2), we have:

$$x_{n+1} \leq x_n \exp(r_1 - a_{11}x_n), \quad n = 0, 1, 2, \ldots.$$

Suppose that x_0 is positive, then using Proposition 2, we obtain:

$$\mathrm{limsup}_{n\to\infty}\, x_n \leq \frac{1}{a_{11}} \exp(r_1 - 1) \coloneqq M_1.$$

The succeeding equation of system (2) yields:

$$\mathrm{limsup}_{n\to\infty}\, y_n \leq \frac{1}{a_{22}} \exp(a_{21}M_1 - 1) \coloneqq M_2$$

whenever $y_0 > 0$. Suppose that $z_0 > 0$. The third equation of system (2) produces:

$$z_{n+1} \leq z_n \exp\left(a_{31}M_1 + \frac{a_{32}}{h} - a_{33}z_n\right).$$

Using again Proposition 2, we have:

$$\mathrm{limsup}_{n\to\infty}\, z_n \leq \frac{1}{a_{33}} \exp\left(\frac{ha_{31}M_1 + a_{32}}{h} - 1\right) \coloneqq M_3.$$

This implies that $\mathrm{limsup}_{n\to\infty}(x_n, y_n, z_n) \leq M$, where $M = \max\{M_1, M_2, M_3\}$. Then, the proof is completed.

Feasibility of Fixed Points

There exist four fixed points for system (2), namely, $E_0 = (0,0,0)$ and $E_1 = \left(\frac{r_1}{a_{11}}, 0, 0\right)$, $E_{12} = (\bar{x}, \bar{y}, 0)$ where:

$$\bar{x} = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \qquad \bar{y} = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}},$$

provided $r_1 a_{21} > r_2 a_{11}$ and $E_{13} = (\hat{x}, 0, \hat{z})$ where:

$$\hat{x} = \frac{r_1 a_{33} + a_{13} r_3}{a_{11}a_{33} + a_{13}a_{31}}, \qquad \hat{z} = \frac{r_1 a_{31} - r_3 a_{11}}{a_{11}a_{33} + a_{13}a_{31}},$$

provided $r_1 a_{31} > r_3 a_{11}$.

The positive fixed point $E^*(x^*, y^*, z^*)$ can be determined from the following equations:

$$\begin{aligned} r_1 - a_{11}x - a_{12}y - a_{13}z &= 0, \\ a_{21}x - a_{22}y - \frac{a_{23}z}{1+hy} - r_2 &= 0, \\ a_{31}x - \frac{a_{32}y}{1+hy} - a_{33}z - r_3 &= 0. \end{aligned} \tag{3}$$

Nature of a Positive Fixed Point

The variational matrix for (2) at E^* is given by:

$$J(x^*, y^*, z^*) = \begin{pmatrix} 1 - a_{11}x^* & -a_{12}x^* & -a_{13}x^* \\ a_{21}y^* & 1 - a_{22}y^* + \frac{a_{23}y^*z^*}{(1+hy^*)^2} & \frac{a_{23}y^*}{1+hy^*} \\ a_{31}z^* & \frac{a_{32}z^*}{(1+hy^*)^2} & 1 - a_{33}z^* \end{pmatrix}.$$

The characteristic polynomial of $J(E^*)$ is given by:

$$P(\lambda) = \lambda^3 + p_1 + p_1\lambda^2 + p_2\lambda + p_3 \tag{4}$$

where:

$$p_1 = a_{11}x^* + a_{22}y^* + a_{33}z^* - 3 - \frac{ha_{23}y^*z^*}{(1+hy^*)^2},$$

$$p_2 = (1 - a_{11}x^*)(2 - a_{22}y^* - a_{33}z^* + \frac{ha_{23}y^*z^*}{(1+hy^*)^2} + (1 - a_{33}z^*)\left(1 - a_{22}y^* + \frac{ha_{23}y^*z^*}{(1+hy^*)^2}\right)$$
$$+ \frac{a_{23}a_{32}y^*z^*}{(1+hy^*)^2} + a_{12}a_{21}x^*y^* + a_{13}a_{31}x^*z^*,$$

$$p_3 = (a_{11}x^* - 1)\{(1 - a_{33}z^*)\left\{(1 - a_{22}y^*) + \frac{ha_{23}y^*z^*}{(1+hy^*)^2} + \frac{a_{23}a_{32}y^*z^*}{(1+hy^*)^3}\right\} + a_{12}a_{21}x^*y^*(a_{33}z^* - 1) - \frac{a_{12}a_{31}a_{23}x^*y^*z^*}{1+hy^*} + a_{13}x^*\left\{\frac{a_{21}a_{32}y^*z^*}{(1+hy^*)^2} - a_{31}z^*\left(1 - a_{22}y^* + \frac{ha_{23}y^*z^*}{(1+hy^*)^2}\right)\right\}.$$

The following proposition is useful to explain the local behaviour of the interior fixed point E^*.

Proposition 3. [33] Define the cubic equation:

$$\lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 = 0 \tag{5}$$

where q_1, q_2, and q_3 are finite. All the roots of Equation (5) remain in an open disk $|\lambda| < 1$ if and only if $|q_1 + q_3| < 1 + q_2, |q_1 - 3q_3| < 3 - q_2$ and $q_3^2 + q_2 - q_3q_1 < 1$.

Proposition 3 helps us to determine the nature of E^*.

Proposition 4. E^* is stable if and only if the inequalities below hold:

$$|p_1 + p_3| < 1 + p_2, |p_1 - 3p_3| < 3 - p_2 \text{ and } p_3^2 + p_2 - p_3p_1 < 1$$

where p_1, p_2, and p_3 are stated after Equation (4).

Global Analysis

Global stability analysis of E^* is based on an iteration scheme and the comparison principle. To prove the global characteristic, we apply the following propositions.

Proposition 5. [34] Suppose that $s(w) = w\exp(\sigma - pw)$, where σ and $p > 0$. Then, $s(w)$ is non-decreasing for $w \in \left(0, \frac{1}{p}\right]$.

Proposition 6. [34] Suppose that the sequence w_n fulfils $w_{n+1} = w_n\exp(\sigma - pw_n)$, $n = 1, 2, 3, \ldots$ where $\sigma, p > 0$. Then,

i. If $\sigma < 2$, then $\lim_{n\to\infty} w_n = \frac{\sigma}{p}$.
ii. If $\sigma \leq 1$, then $w_n \leq \frac{1}{p}, n = 2, 3, \ldots$.

Proposition 7. [35] Assume that functions $s, g : \mathbb{Z}_+ \times [0, \infty)$ satisfy $s(n, x) \leq g(n, x)$ $\left(s(n, x) \geq g(n, x)\right)$ for $n \in \mathbb{Z}_+$ and $g(n, x)$ is non-decreasing with respect to x. If w_n satisfies the following equations:

$$x_{n+1} = s(n, x_n), w_{n+1} = g(n, u_n)$$

respectively, and $x_0 \leq w_0$ $(x_0 \geq w_0)$ then $x_n \leq w_n$ $(x_n \geq w_n)$ for all $n \geq 0$.

In this section, we demonstrate the positivity and boundedness of solutions to system (2). The proof of the first proposition is trivial, and its evidence is not given.

Theorem 2. Assume that $r_1 \leq 1,\ a_{21}(P_1^x + \varepsilon) \leq 1 + r_2$ and $\frac{a_{31}(P_1^x+\varepsilon)\left(1+h(P_1^y+\varepsilon)\right)+a_{32}(P_1^y+\varepsilon)}{1+h(P_1^y+\varepsilon)} \leq 1 + r_3$, then E^* of system (2) is globally asymptotically stable.

Proof. For any solution (x_n, y_n, z_n) of system (2) with positive initial values (x_0, y_0, z_0), define:

$$X_1 = \mathrm{limsup}_{n\to\infty}\, x_n\,, Y_1 = \mathrm{liminf}_{n\to\infty}\, x_n,$$

$$X_2 = \mathrm{limsup}_{n\to\infty}\, y_n\,, Y_2 = \mathrm{liminf}_{n\to\infty}\, y_n,$$

$$X_3 = \mathrm{limsup}_{n\to\infty}\, z_n\,, Y_3 = \mathrm{liminf}_{n\to\infty}\, z_n.$$

We now show that $X_1 = Y_1 = x^*,\ X_2 = Y_2 = y^*,\ X_3 = Y_3 = z^*$. First, we show that $X_1 \leq P_1^x, X_2 \leq P_1^y, X_3 \leq P_1^z$. The first equation of system (2) yields:

$$x_{n+1} \leq x_n \exp(r_1 - a_{11}x_n)\,,\ \ n = 0, 1, 2, \ldots .$$

Taking the auxiliary equation:

$$w_{n+1} = w_n \exp(\mathrm{r}_1 - \mathrm{a}_{11}\mathrm{w}_\mathrm{n})\,, \tag{6}$$

by Proposition 6 (ii), as $r_1 \leq 1$, we find $w_n \leq \frac{1}{a_{11}}$ for all $n \geq 2$. Using Proposition 5, we have $s(w) = w \exp(\sigma - a_{11}w)$ is non-decreasing for $w \in \left(0, \frac{1}{a_{11}}\right]$. Proposition 7 produces $x_n \leq w_n$ for all $n \geq 2$, where w_n is the solution of Equation (6) with initial value $w_2 = x_2$. Applying Proposition 6 (i), we have:

$$X_1 = \mathrm{limsup}_{n\to\infty}\, x_n \leq \lim_{n\to\infty} w_n = \frac{r_1}{a_{11}} \triangleq P_1^x.$$

Hence, given any $\epsilon > 0$, there exists a $n_1 > 2$ for which if $n \geq n_1$, then $x_n \leq P_1^x + \epsilon$. The second equation of system (2) gives:

$$y_{n+1} \leq y_n \exp(a_{21}(P_1^x + \varepsilon) - a_{22}y_n - r_2)\,, n = \ 0, 1, 2, \ldots.$$

Taking the auxiliary equation:

$$w_{n+1} = w_n \exp(a_{21}(P_1^x + \varepsilon) - a_{22}w_n - r_2)\,, \tag{7}$$

by Proposition 6 (ii), as $a_{21}(P_1^x + \varepsilon) \leq 1 + r_2$, we get $w_n \leq \frac{1}{a_{22}}$ for all $n \geq 2$. By Proposition 5, we have $s(w) = \mathrm{w} \exp(a_{21}(P_1^x + \varepsilon) - a_{22}w - r_2)$ is non-decreasing for $w \in \left(0, \frac{1}{a_{22}}\right]$. Using Proposition 7, we see $y_n \leq w_n$ for all $n \geq 2$, where w_n is the solution of Equation (6) with initial value $w_2 = x_2$. Applying Proposition 6 (i), we find:

$$X_2 = \mathrm{limsup}_{n\to\infty}\, y_n \leq \lim_{n\to\infty} w_n = \frac{a_{21}(P_1^x + \varepsilon) - r_2}{a_{22}} \triangleq P_1^y.$$

Thus, for any $\epsilon > 0$, one can get a $n_2 > n_1$ for which if $n \geq n_2$, then $y_n \leq P_1^y + \epsilon$. The third equation of system (2) produces:

$$z_{n+1} \leq z_n \exp\left\{\frac{a_{31}(P_1^x + \varepsilon)\left(1 + h\left(P_1^y + \varepsilon\right)\right) + a_{32}\left(P_1^y + \varepsilon\right)}{1 + h\left(P_1^y + \varepsilon\right)} - r_3 - a_{33}z_n\right\}.$$

Further, taking the auxiliary equation:

$$w_{n+1} \leq w_n \exp\left\{\frac{a_{31}(P_1^x + \varepsilon)\left(1 + h\left(P_1^y + \varepsilon\right)\right) + a_{32}\left(P_1^y + \varepsilon\right)}{1 + h\left(P_1^y + \varepsilon\right)} - r_3 - a_{33}w_n\right\}.$$

Using Proposition 6 (ii), as $\frac{a_{31}(P_1^x+\varepsilon)\left(1+h\left(P_1^y+\varepsilon\right)\right)+a_{32}\left(P_1^y+\varepsilon\right)}{1+h\left(P_1^y+\varepsilon\right)} - r_3 \leq 1$, we have $w_n \leq \frac{1}{a_{33}}$ for all $n \geq 2$. From Proposition 5, it follows that $s(w) = w \exp\left\{\frac{a_{31}(P_1^x+\varepsilon)\left(1+h\left(P_1^y+\varepsilon\right)\right)+a_{32}\left(P_1^y+\varepsilon\right)}{1+h\left(P_1^y+\varepsilon\right)} - r_3 - a_{33}w\right\}$ is non-decreasing for $w \in \left(0, \frac{1}{a_{33}}\right]$. Proposition 7 yields that $z_n \leq w_n$ for all $n \geq 2$. Accordingly,

$$X_3 = \mathrm{limsup}_{n\to\infty}\, z_n \leq \lim_{n\to\infty} w_n = \frac{1}{a_{33}}\left[\frac{a_{31}(P_1^x + \varepsilon)\left(1 + h\left(P_1^y + \varepsilon\right)\right) + a_{32}\left(P_1^y + \varepsilon\right)}{1 + h\left(P_1^y + \varepsilon\right)} - r_3\right] \triangleq P_1^z.$$

So, for any $\epsilon > 0$, one can find $n_3 > n_2$ for which $n \geq n_3$, $z_n \leq P_1^z + \epsilon$.

Now we prove that $Y_1 \geq Q_1^x, Y_2 \geq Q_1^y, Y_3 \geq Q_1^z$. Using the first equation of system (2), one can get:

$$x_{n+1} \geq x_n \exp\left[r_1 - a_{11}x_n - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon)\right], \mathrm{n} \geq \mathrm{n}_3.$$

As shown in the auxiliary equation:

$$w_{n+1} \geq w_n \exp\left[r_1 - a_{11}w_n - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon)\right]. \tag{8}$$

As $r_1 - a_{11}x_n - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon) < 1$, using Proposition 6 (ii), we get $w_n \leq \frac{1}{a_{11}}$ for $n \geq n_3$. It follows from Proposition 5 that $f(w) = w \exp\left[r_1 - a_{11}w - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon)\right]$ is non-decreasing for $w \in \left(0, \frac{1}{a_{11}}\right]$. Thus, from Proposition 7, we get $x_n \geq w_n$ for all $n \geq n_3$. Applying Proposition 6 (i), we find:

$$Y_1 = \mathrm{liminf}_{n\to\infty}\, x_n \geq \lim_{n\to\infty} w_n = \frac{r_1 - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon)}{a_{11}}.$$

As $\epsilon > 0$ is arbitrary, we get:

$$Y_1 \geq Q_1^x = \frac{r_1 - a_{12}\left(P_1^y + \epsilon\right) - a_{13}(P_1^z + \epsilon)}{a_{11}}.$$

Thus, for any $\epsilon > 0$, one can find $n_4 > n_3$ for which $n \geq n_4$, $x_n \geq Q_1^x - \epsilon$. The second equation of system (2) produces:

$$y_{n+1} \geq y_n \exp\left[a_{21}(Q_1^x - \epsilon) - a_{23}(P_1^z + \epsilon) - r_2 - a_{22}y_n\right], n \geq n_4.$$

Using similar reasoning, we have:

$$Y_2 = \mathrm{liminf}_{n\to\infty}\, y_n \geq \lim_{n\to\infty} w_n = \frac{a_{21}(Q_1^x - \epsilon) - a_{23}(P_1^z + \epsilon) - r_2}{a_{22}}.$$

As $\epsilon > 0$ is arbitrary, we have:

$$Y_2 \geq Q_1^y = \frac{a_{21}(Q_1^x - \epsilon) - a_{23}(P_1^z + \epsilon) - r_2}{a_{22}}.$$

Thus, for any $\epsilon > 0$, one can find $n_5 > n_4$ for which $n \geq n_5$, $y_n \geq Q_1^y - \epsilon$. The third equation of system (2) yields:

$$z_{n+1} \geq z_n \exp\left[a_{31}(Q_1^x - \epsilon) + \frac{a_{32}(Q_1^y - \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_3 - a_{33} z_n\right], n \geq n_5$$

which implies that:

$$Y_3 = \text{liminf}_{n\to\infty} z_n \geq \lim_{n\to\infty} w_n = \frac{1}{a_{33}}\left[a_{31}(Q_1^x - \epsilon) + \frac{a_{32}(Q_1^y - \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_3\right].$$

As $\epsilon > 0$ is arbitrary, we find:

$$Y_3 \geq Q_1^z = \frac{1}{a_{33}}\left[a_{31}(Q_1^x - \epsilon) + \frac{a_{32}(Q_1^y - \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_3\right].$$

Thus, for any $\epsilon > 0$, one can find $n_6 > n_5$ for which $n \geq n_6$, $z_n \geq Q_1^z - \epsilon$.

Our next task is to prove that $X_1 \leq P_2^x$, $X_2 \leq P_2^y$ and $X_3 \leq P_2^z$, where $P_2^x \leq P_1^x$, $P_2^y \leq P_1^y$ and $P_2^z \leq P_1^z$, respectively. For $n > n_6$, the first equation of system (2) gives:

$$x_{n+1} \leq x_n \exp\left[r_1 - a_{11} x_n - a_{12}(Q_1^y - \epsilon) - a_{13}(Q_1^z - \epsilon)\right].$$

Let the auxiliary equation be:

$$w_{n+1} = w_n \exp\left[r_1 - a_{11} w_n - a_{12}(Q_1^y - \epsilon) - a_{13}(Q_1^z - \epsilon)\right]. \tag{9}$$

Proceeding along the previous reasoning, we obtain:

$$X_1 = \text{limsup}_{n\to\infty} x_n \leq \frac{1}{a_{11}}\left[r_1 - a_{12}(Q_1^y - \epsilon) - a_{13}(Q_1^z - \epsilon)\right].$$

Since, $r_1 - a_{12}(Q_1^y - \epsilon) - a_{13}(Q_1^z - \epsilon) \leq 1$. As $\epsilon > 0$ is arbitrary, we assert that:

$$X_1 \leq P_2^x = \frac{1}{a_{11}}\left[r_1 - a_{12}(Q_1^y - \epsilon) - a_{13}(Q_1^z - \epsilon)\right].$$

Consequently, for any $\epsilon > 0$, one can find $n_7 > n_6$ so that for $n \geq n_7, x_n \leq P_2^x + \epsilon$. Likewise, from the second equation of system (2) for $n > n_7$, we have:

$$y_{n+1} \leq y_n \exp\left[a_{21}(P_2^x + \epsilon) - a_{22} y_n - \frac{a_{23}(Q_1^z - \epsilon)}{1 + h(P_1^y + \epsilon)} - r_2\right].$$

From previous reasoning, we derive:

$$X_2 \leq P_2^y = \frac{1}{a_{22}}\left[a_{21}(P_2^x + \epsilon) - \frac{a_{23}(Q_1^z - \epsilon)}{1 + h(P_1^y + \epsilon)} - r_2\right].$$

Consequently, for any $\epsilon > 0$, one can find $n_8 > n_7$ so that for $n \geq n_8, y_n \leq P_2^y + \epsilon$. For $n > n_8$, the third equation of system (2) produces:

$$z_{n+1} \leq z_n \exp\left[a_{31}(P_2^x + \epsilon) - a_{33}z_n + \frac{a_{32}(P_2^y + \epsilon)}{1 + h(P_2^y + \epsilon)} - r_3\right].$$

Following the previous reasoning, we have:

$$X_3 \leq P_2^z = \frac{1}{a_{33}}\left[a_{31}(P_2^x + \epsilon) + \frac{a_{32}(P_2^y + \epsilon)}{1 + h(P_2^y + \epsilon)} - r_3\right].$$

Thus, for any $\epsilon > 0$ one can find $n_9 > n_8$ so that for $n \geq n_9, z_n \leq P_2^x + \epsilon$.

Subsequently, we prove that $Y_1 \geq Q_2^x, Y_2 \geq Q_2^y$, and $Y_3 \geq Q_2^z$, where $Q_2^x \geq Q_1^x, Q_2^y \geq Q_1^y$, and $Q_2^z \geq Q_1^z$, respectively. Again, from the first system (2) for $n > n_9$, we derive:

$$x_{n+1} \geq x_n \exp\left[r_1 - a_{11}x_n - a_{12}(P_2^y + \epsilon) - a_{13}(P_2^z + \epsilon)\right].$$

Proceeding along a similar reasoning, we have:

$$Y_1 = \text{liminf}_{n\to\infty}\, x_n \geq \frac{1}{a_{11}}\left[r_1 - a_{12}(P_2^y + \epsilon) - a_{13}(P_2^z + \epsilon)\right]\cdot$$

Since $r_1 - a_{12}(P_2^y + \epsilon) - a_{13}(P_2^z + \epsilon) \leq 1$. As $\epsilon > 0$ is arbitrary, we assert that:

$$Y_1 \geq Q_2^x = \frac{1}{a_{11}}\left[r_1 - a_{12}(P_2^y + \epsilon) - a_{13}(P_2^z + \epsilon)\right].$$

So, for any $\epsilon > 0$ one can find $n_{10} > n_9$ so that for $n \geq n_{10}$, $x_n \geq Q_2^x - \epsilon$. For $n \geq n_{10}$, the second equation of system (2) gives:

$$y_{n+1} \geq y_n \exp\left[a_{21}(Q_1^x - \epsilon) - a_{22}y_n - \frac{a_{23}(P_2^z + \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_2\right]$$

which implies that:

$$Y_2 = \text{liminf}_{n\to\infty}\, y_n \geq \frac{1}{a_{22}}\left[a_{21}(Q_1^x - \epsilon) - \frac{a_{23}(P_2^z + \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_2\right]\cdot$$

As $\epsilon > 0$ is arbitrary, we assert that:

$$Y_2 \geq Q_2^y = \frac{1}{a_{22}}\left[a_{21}(Q_1^x - \epsilon) - \frac{a_{23}(P_2^z + \epsilon)}{1 + h(Q_1^y - \epsilon)} - r_2\right].$$

So, for any $\epsilon > 0$ one can find $n_{11} > n_{10}$ so that for $n \geq n_{11}$, $y_n \geq Q_2^y - \epsilon$. For $n \geq n_{11}$, the third equation of system (2) yields:

$$z_{n+1} \geq z_n \exp\left[a_{31}(Q_2^x - \epsilon) - a_{33}z_n + \frac{a_{32}(Q_2^y - \epsilon)}{1 + h(Q_2^y - \epsilon)} - r_3\right]$$

which implies that:

$$Y_3 = \text{liminf}_{n\to\infty} z_n \geq \frac{1}{a_{33}}\left[a_{31}(Q_2^x - \epsilon) + \frac{a_{32}\left(Q_2^y - \epsilon\right)}{1 + h\left(Q_2^y - \epsilon\right)} - r_3\right].$$

As $\epsilon > 0$ is arbitrary, we assert that:

$$Y_3 \geq Q_2^z = \frac{1}{a_{33}}\left[a_{31}(Q_2^x - \epsilon) + \frac{a_{32}\left(Q_2^y - \epsilon\right)}{1 + h\left(Q_2^y - \epsilon\right)} - r_3\right].$$

So, for any $\epsilon > 0$ one can find $n_{12} > n_{11}$ so that for $n \geq n_{12}, z_n \geq Q_2^z - \epsilon$.

Performing the above operation repeatedly, we finally obtain six sequences $\{P_n^x\}, \{P_n^y\}, \{P_n^z\}, \{Q_n^x\}, \{Q_n^y\}$, and $\{Q_n^z\}$ such that for all $n \geq 2$,

$$\begin{aligned}
P_n^x &= \frac{1}{a_{11}}\left[r_1 - a_{12}Q_{n-1}^y - a_{13}Q_{n-1}^z\right],\\
P_n^y &= \frac{1}{a_{22}}\left[a_{21}P_n^x - \frac{a_{23}Q_{n-1}^z}{1 + hP_{n-1}^y} - r_2\right],\\
P_n^z &= \frac{1}{a_{33}}\left[a_{31}P_n^x + \frac{a_{32}P_n^y}{1 + hP_n^y} - r_3\right],\\
Q_n^x &= \frac{1}{a_{11}}\left[r_1 - a_{12}P_n^y - a_{13}P_n^z\right],\\
Q_n^y &= \frac{1}{a_{22}}\left[a_{21}Q_{n-1}^x - \frac{a_{23}P_n^z}{1 + hQ_{n-1}^y} - r_2\right],\\
Q_n^z &= \frac{1}{a_{33}}\left[a_{31}Q_n^x + \frac{a_{32}Q_n^y}{1 + hQ_n^y} - r_3\right].
\end{aligned} \tag{10}$$

It is evident that for any integer $n > 0$,

$$Q_n^x \leq Y_1 \leq X_1 \leq P_n^x, Q_n^y \leq Y_2 \leq X_2 \leq P_n^y, Q_n^z \leq Y_3 \leq X_3 \leq P_n^z.$$

Now, we will show that $\{P_n^x\}, \{P_n^y\}$, and $\{P_n^z\}$ are monotonically decreasing and $\{Q_n^x\}, \{Q_n^y\}$, and $\{Q_n^z\}$ are monotonically increasing, by the application of the induction theory. At first, it is obvious that $P_2^x \leq P_1^x, P_2^y \leq P_1^y, P_2^z \leq P_1^z, Q_2^x \geq Q_1^x, Q_2^y \geq Q_1^y$, and $Q_2^z \geq Q_1^z$. For $n = k$ $(k \geq 2)$, we assume that $P_k^x \leq P_{k-1}^x, P_k^y \leq P_{k-1}^y, P_k^z \leq P_{k-1}^z, Q_k^x \geq Q_{k-1}^x, Q_k^y \geq Q_{k-1}^y$, and $Q_k^z \geq Q_{k-1}^z$. Now,

$$P_{k+1}^x - P_k^x = -\frac{1}{a_{11}}\left[a_{12}\left(Q_k^y - Q_{k-1}^y\right) + a_{13}(Q_k^z - Q_{k-1}^z)\right] \leq 0,$$
$$P_{k+1}^y - P_k^y = \frac{1}{a_{22}}\left[a_{21}(P_{k+1}^x - P_k^x) - a_{23}\left\{\frac{Q_{k-1}^z - Q_k^z + h\left(P_k^y Q_{k-1}^z - P_{k-1}^y Q_k^z\right)}{\left(1 + hP_{k-1}^y\right)\left(1 + hP_k^y\right)}\right\}\right] \leq 0,$$
$$P_{k+1}^z - P_k^z = \frac{1}{a_{33}}\left[a_{31}(P_{k+1}^x - P_k^x) + \frac{a_{32}\left(P_{k+1}^y - P_k^y\right)}{\left(1 + hP_{k+1}^y\right)\left(1 + hP_k^y\right)}\right] \leq 0,$$
$$Q_{k+1}^x - Q_k^x = -\frac{1}{a_{11}}\left[a_{12}\left(P_{k+1}^y - P_k^y\right) + a_{13}(P_{k+1}^z - P_k^z)\right] \geq 0,$$
$$Q_{k+1}^y - Q_k^y = \frac{1}{a_{22}}\left[a_{21}(Q_k^x - Q_{k-1}^x) + a_{23}\left\{\frac{P_k^z - Q_{k+1}^z + h\left(P_k^z Q_k^y - P_{k+1}^z Q_{k-1}^y\right)}{\left(1 + hQ_k^y\right)\left(1 + hQ_{k-1}^y\right)}\right\}\right] \geq 0,$$
$$Q_{k+1}^z - Q_k^z = \frac{1}{a_{33}}\left[a_{31}(Q_{k+1}^x - Q_k^x) + \frac{a_{32}\left(Q_{k+1}^y - Q_k^y\right)}{\left(1 + hQ_{k+1}^y\right)\left(1 + hQ_k^y\right)}\right] \geq 0.$$

This establishes that $\{P_n^x\}, \{P_n^y\}$, and $\{P_n^z\}$ are monotonically decreasing and $\{Q_n^x\}, \{Q_n^y\}$, and $\{Q_n^z\}$ are monotonically increasing. Due to the monotonicity, it can be proven that these sequences have a limit.

Let $\lim_{n\to\infty} P_n^x = x_1, \lim_{n\to\infty} P_n^y = x_2, \lim_{n\to\infty} P_n^z = x_3, \lim_{n\to\infty} Q_n^x = y_1, \lim_{n\to\infty} Q_n^y = y_2$, and $\lim_{n\to\infty} Q_n^z = y_3$. Taking limit as $n \to \infty$ in (10), one can obtain:

$$\begin{aligned}
x_1 &= \frac{1}{a_{11}}[r_1 - a_{12}y_2 - a_{13}y_3],\\
x_2 &= \frac{1}{a_{22}}\left[a_{21}x_1 - \frac{a_{23}y_3}{1 + hx_2} - r_2\right],\\
x_3 &= \frac{1}{a_{33}}\left[a_{31}x_1 + \frac{a_{32}x_2}{1 + hx_2} - r_3\right],\\
y_1 &= \frac{1}{a_{11}}[r_1 - a_{12}x_2 - a_{13}x_3],\\
y_3 &= \frac{1}{a_{33}}\left[a_{31}y_1 + \frac{a_{32}y_2}{1 + hy_2} - r_3\right],\\
y_3 &= \frac{1}{a_{33}}\left[a_{31}y_1 + \frac{a_{32}y_2}{1 + hy_2} - r_3\right].
\end{aligned} \tag{11}$$

We note that $x_1 = y_1, x_2 = y_2$ and $x_3 = y_3$. Hence, we have $x_1 = x^*, x_2 = y^*, x_3 = z^*$ as a solution of (11). Thus, the global asymptotic stability of E^* is established and this ends the proof.

Bifurcation Investigation

In this part, we will derive the conditions for determining NS and PD bifurcation at the positive fixed point E^* of system (2).

Neimark-Sacker Bifurcation

To study NS bifurcation in system (2), we require the proposition [36] stated below.

Proposition 8. Let an n-dimensional system be $V_{k+1} = g_m(V_k)$ considering $m \in \mathbb{R}$ as a bifurcation parameter. Let V^* be a fixed point of g_m and the characteristic polynomial for the variational matrix $J(V^*) = (b_{ij})_{n\times n}$ of $g_m(V_k)$ is described by:

$$P_m(\lambda) = \lambda^n + b_1\lambda^{n-1} + \cdots + b_{n-1}\lambda + b_n \tag{12}$$

with $b_i = b_i(m, v), i = 1, 2, 3, ..., n$ and v is a parameter which controls the system. The sequence of determinants $\Delta_0^{\pm}(m, v) = 1, \Delta_1^{\pm}(m, v), \cdots, \Delta_n^{\pm}(m, v)$ designated by $\Delta_i^{\pm}(m, v) = \det(M_1 \pm M_2),\ i = 1,2,3,\cdots,n$ where

$$M_1 = \begin{pmatrix} 1 & b_1 & b_2 & \cdots & b_{i-1} \\ 0 & 1 & b_1 & \cdots & b_{i-2} \\ 0 & 0 & 1 & \cdots & b_{i-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, M_2 = \begin{pmatrix} b_{n-i+1} & b_{n-i+2} & \cdots & b_{n-1} & b_n \\ b_{n-i+2} & b_{n-i+3} & \cdots & b_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_n & \cdots & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Furthermore, the following assumptions hold:

- **A1.** $\Delta_{n-1}^{-}(m_0, v) = 0, \Delta_{n-1}^{+}(m_0, v) > 0,\ \ P_{m_0}(1) > 0, (-1)^n P_{m_0}(-1) > 0,\ \ \Delta_i^{+}(m_0, v) > 0, i = n-3, n-5, \cdots, 1$ (or 2), when n is even or odd, respectively.
- **A2.** $\left[\frac{d\left(\Delta_{n-1}^{-}(m,v)\right)}{dm}\right]_{m=m_0} \neq 0.$
- **A3.** $\cos\left(\frac{2\pi}{j}\right) \neq \psi$, or $\cos\left(\frac{2\pi}{j}\right) = \psi$ *where* $j = 3, 4, 5, \cdots$ *and* $\psi = 1 - \frac{0.5P_{m_0}(1)\Delta_{n-3}^{-}(m_0,v)}{\Delta_{n-2}^{+}(m_0,v)}$.

Then, NS bifurcation emerges at m_0.

Now we state a bifurcation result on parameter r_1.

Theorem 3. The fixed point E^* of system (2) allows NS bifurcation when the conditions stated below are fulfilled:

$$\begin{aligned} 1 - p_2 + p_3(p_1 - p_3) &= 0, \\ 1 + p_2 - p_3(p_1 + p_3) &> 0, \\ 1 + p_1 + p_2 + p_3 &> 0, \\ 1 - p_1 + p_2 - p_3 &> 0 \end{aligned} \tag{13}$$

where p_1, p_2, and p_3 are defined after Equation (4).

Proof. Using Proposition 5, we get:

$$\begin{aligned} \Delta_2^{-}(r_1^*) &= 1 - p_2 + p_3(p_1 - p_3) = 0, \\ \Delta_2^{+}(r_1^*) &= 1 + p_2 - p_3(p_1 + p_3) > 0, \\ P_{r_1^*}(1) &= 1 + p_1 + p_2 + p_3 > 0, \\ (-1)^3 P_{r_1^*}(1) &= 1 - p_1 + p_2 - p_3 > 0. \end{aligned} \tag{14}$$

Period Doubling Bifurcation

We check whether the system (2) possesses a PD bifurcation or not. To examine the above bifurcation, we use the proposition stated below.

Proposition 9. [37] Suppose the restrictions excepting **A1**, **A2**, and **A3** in Proposition 8 are satisfied.

Moreover, the following conditions are fulfilled:

H1. $P_{m_0}(-1) = 0,\ \Delta^{\pm}_{n-1}(m_0, v) > 0, \Delta^{\pm}_{i}(m_0, v) > 0, i = n-2, n-4, \cdots, 1$ (or 2), when n is even or odd, respectively.

H2. $\frac{\sum_{i=1}^{n}(-1)^{n-i} b_i'}{\sum_{i=1}^{n}(-1)^{n-i}(n-i+1)b_{i-1}} \neq 0$ where b_i' represents the derivative of $b_i(m)$ *at* $m = m_0$.

Then a period-doubling bifurcation occurs at m_0.

Theorem 4. The fixed point E^* of system (2) allows a period doubling bifurcation at $a_{11} = a_{11}^*$ if the following conditions are satisfied:

$$
\begin{aligned}
1 - p_2 + p_3(p_1 - p_3) &> 0,\\
1 + p_2 - p_3(p_1 + p_3) &> 0,\\
1 + p_1 + p_2 + p_3 &> 0,\\
1 - p_1 + p_2 - p_3 &= 0,\\
1 \pm p_3 &> 0,\\
\frac{p_1' - p_2' + p_3'}{3 - 2p_1 + p_2} &\neq 0,
\end{aligned}
$$

where p_1, p_2, and p_3 are defined after Equation (4) and p_i' stands for the derivative of $p_i(a_{11})$ with respect to a_{11} at $a_{11} = a_{11}^*$.

Proof. From Proposition 9, one can easily prove the theorem.

Chaos Control

Now, we investigate how the chaotic behaviour of system (2) can be regulated. This is also reasonable for interacting populations. It is a natural fact that chaotic orbits are found in discrete-time models rather than continuous ones. Thus, it is justifiable to employ control measures to regulate any unwanted situations. Inspired by [38], we use a hybrid control method. A single control parameter is sufficient to utilise this method. There are several ways to prevent chaos in discrete models, such as hybrid control, state feedback, and pole-placement methods [39, 40, 41]. For mathematical simplicity, we follow a hybrid control method to system (2) for preventing chaos when bifurcation arises in the system. Suppose that the NS and PD bifurcation at E^* then using the hybrid control method, system (2) becomes:

$$
\begin{aligned}
x_{n+1} &= \mu x_n \exp\{r_1 - a_{11}x_n - a_{12}y_n - a_{13}z_n\} + (1-\mu)x_n,\\
y_{n+1} &= \mu y_n \exp\left\{a_{21}x_n - a_{22}y_n - \frac{a_{23}z_n}{1+hy_n} - r_2\right\} + (1-\mu)y_n,\\
z_{n+1} &= \mu z_n \exp\left\{a_{31}x_n - \frac{a_{32}y_n}{1+hy_n} - a_{33}z_n - r_3\right\} + (1-\mu)z_n.
\end{aligned}
\tag{15}
$$

where $0 < \mu < 1$ is the controlling parameter. The variational matrix of system (15) at E^* is

$$J(x^*, y^*, z^*) = \begin{pmatrix} 1-\mu a_{11}x^* & -\mu a_{12}x^* & -\mu a_{13}x^* \\ \mu a_{21}y^* & 1-\mu\left(a_{22}y^* + \frac{a_{23}z^*h}{(1+hy^*)^2}\right) & -\frac{\mu a_{23}y^*}{1+hy^*} \\ \mu a_{31}z^* & \frac{\mu a_{32}z^*}{(1+hy^*)^2} & 1-\mu a_{33}z^* \end{pmatrix}. \quad (16)$$

The fixed point E^* of system (15) is a sink and locally asymptotically stable if all the roots of the characteristic polynomial of (16) lie in a unit open disk.

Numerical Simulations

Here, we illustrate our analytical findings through numerical simulations. We discuss the effect of the rate of intrinsic growth of and the interspecific competition rate on system (2) dynamics.

Example 1. In system (2), suppose $r_1 = 1, r_2 = 0.5, r_3 = 0.5, a_{11} = 1, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 1, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 1.8$, which fulfils the restrictions of Theorem 2. Thus, all the solutions converge to $E^* = (0{,}8309, 1.581, 0.1104)$ (Figure 1). From Figure 1, we note that at the beginning, each population becomes smaller and in due course all the species obtain the equilibrium states and ultimately global convergence of is ensured.

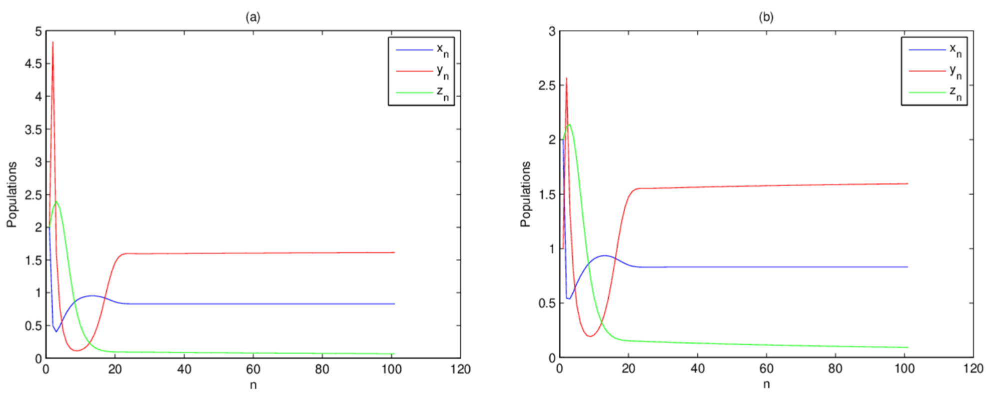

Figure 1: Time series solutions of system (2) where the values of the parameter are taken as $r_1 = 1, r_2 = 0.5, r_3 = 0.5, a_{11} = 1, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 1, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 1.8$ and initial points (2, 2, 2) and (2, 1, 2)

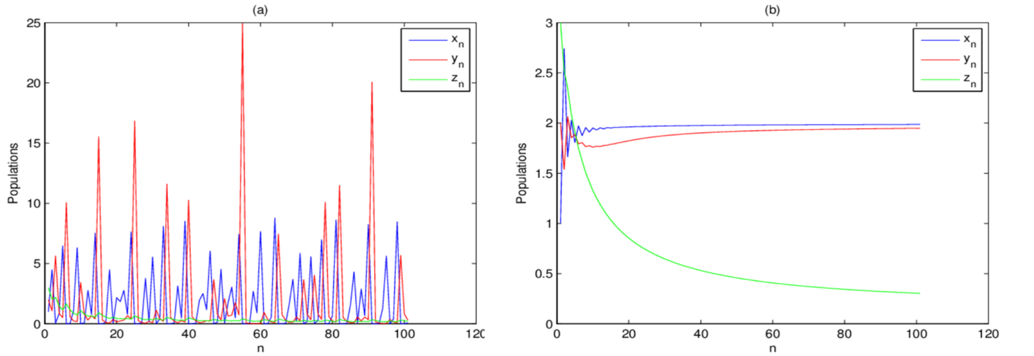

Figure 2: (a) Time series solutions of system (2) where the values of parameters are taken as $r_1 = 3.8, r_2 = 0.8, r_3 = 0.2, a_{11} = 1.8, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 0.6, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 45$, with initial points (1, 2, 3) and (b) phase diagram of system (15) for $\mu = 0.5$

Example 2. Suppose $r_1 = 3.8, r_2 = 0.8, r_3 = 0.2, a_{11} = 1.8, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 0.6, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 45$, and initial points (1, 2, 3) for system (1) which disobey the constraints in Proposition 4. Thus, the instability of $E^* = (1.857, 1.525, 3.045)$. Furthermore, we observe a chaotic nature [Figure 2(a)]. To avoid this phenomenon for system (15), we select $\mu = 0.5$. For this selection of parameters, the fixed point E^* becomes stable. Figure 2 (b) confirms the stability of $E^* = (1.857, 1.525, 3.045)$ of the system (15).

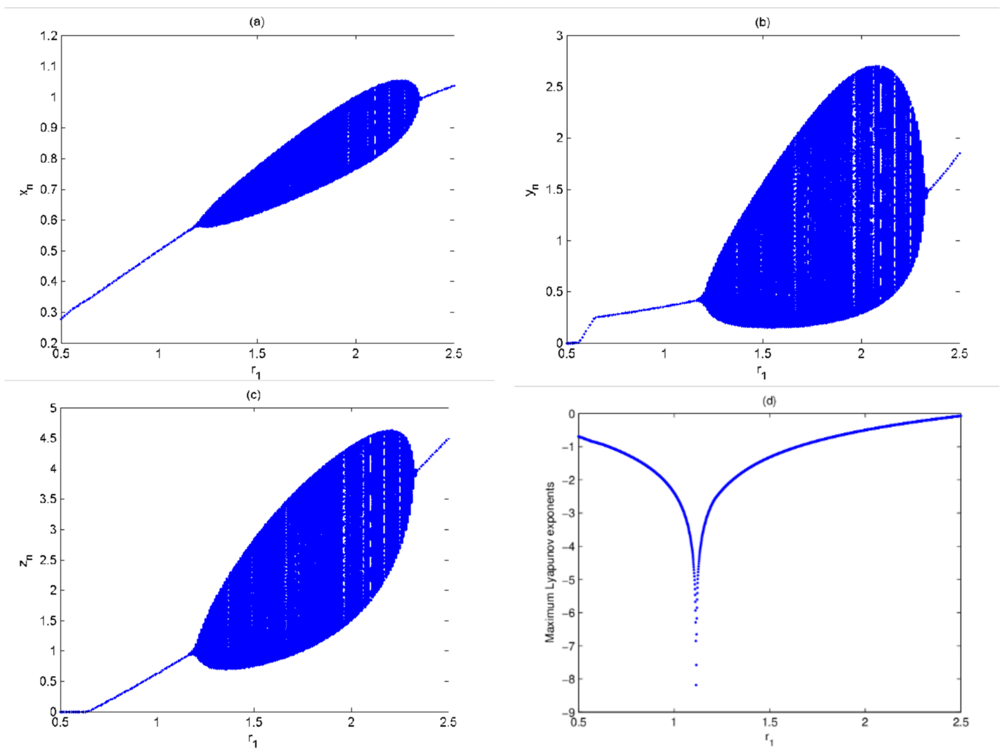

Figure 3: Bifurcation diagrams and MLE for system (2) with respect to r_1

Example 3. Suppose $r_2 = 0.5, r_3 = 0.2, a_{11} = 1.8, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 1.6, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 1$, and initial points $(0.5, 0.5, 0.5)$ and $r_1 \in (0.5, 2.5)$ in system (2). When r_1 is considered as a bifurcation parameter, then, at $r_1 = r_1^* = 1.17$, $E^* = (0.0786, 0.1682, 21.73)$, cannot be stable and system (2) allows NS bifurcation.

The constraints in Theorem 3 are satisfied at $E^* = (0.0786, 0.1682, 21.73)$ for a particular value of the parameter $r_1 = 1.17$. Figure 3 shows bifurcation diagrams and Maximum Lyapunov Exponents (MLE) with respect to the parameter r_1 of system (2). As r_1 get larger, we find a change from stability to instability and then bifurcation within a limit cycle to a periodic window and ultimately to stability.

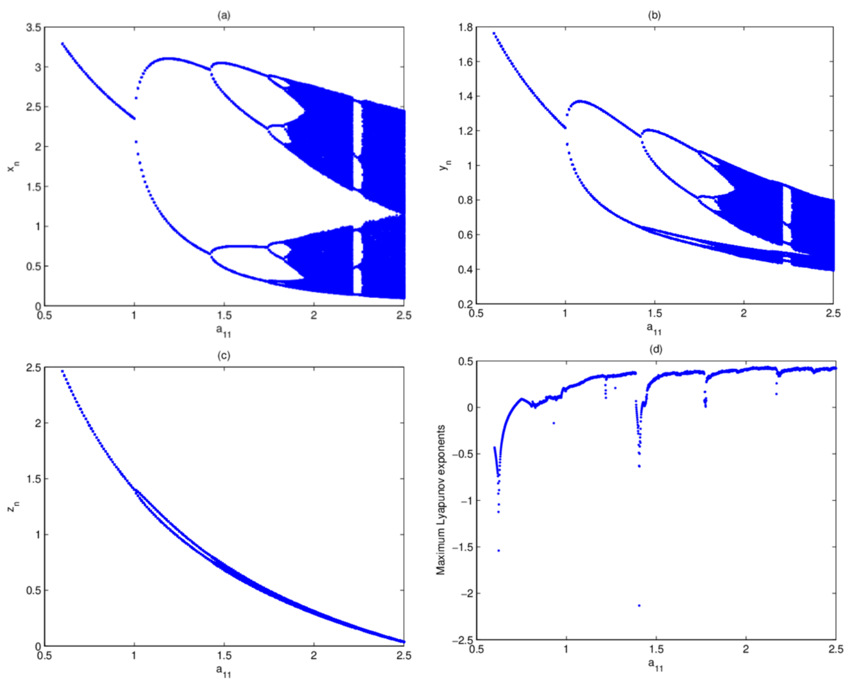

Figure 4: Bifurcation diagrams and MLE for system (2) with respect to a_{11}

Example 4. Suppose $r_1 = 3.1, r_2 = 0.5, r_3 = 0.1, a_{12} = 0.5, a_{13} = 0.1, a_{21} = 0.5, a_{22} = 0.1, a_{23} = 0.4, a_{31} = 0.1, a_{32} = 0.2, a_{33} = 0.1, h = 0.01$, $a_{11} \in (0.6, 2.5)$, and initial points $(0.5, 0.5, 0.5)$. We observe PD bifurcation at $a_{11} = 0.98$, Figure 4 exhibits bifurcation diagrams and MLE with respect to the parameter a_{11} of system (2). From Figure 4, one can see that the stability of the system is stable whenever $a_{11} < 0.98$ and as a_{11} get larger, a series of PD bifurcations in which a 2^k-cycle cannot maintain its stability.

Discussions

In this article, we present and analyse a discrete-time predator-prey system with intraguild predation. The model (2) is formulated based on system (1), which was studied in [31]. To our understanding, there are limited works that address the discrete version of the continuous model under study. We have examined the local and global behaviour of a positive fixed point. Furthermore, NS and

PD bifurcation are investigated. To prevent chaotic orbits, control measure is taken. The system parameter r_1, representing the intrinsic growth rate for prey species x plays a vital role in obtaining NS bifurcation for a certain range of intervals. Again, PD bifurcation may arise for the parameter a_{11}, representing the intraspecific competition among prey species x. These findings concur with those in [31]. The global stability of the positive fixed point is demonstrated using the iteration scheme and the comparison principle. From Theorem 2, one can ensure global stability of the positive fixed point as long as the rate of intrinsic growth of the prey remains below one and the death rates of the intermediate and top predators are high enough. Recently, Blé *et al*. [42] discussed the dynamics of a discrete system arising from an intraguild food web model by applying the average model. Numerical simulations also support our findings.

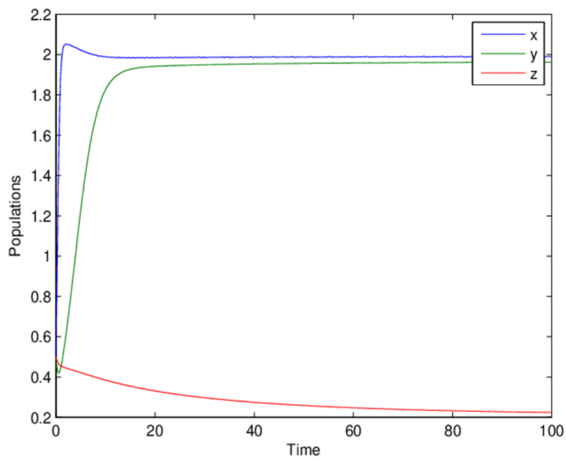

Figure 5: Time series solutions of system (1) where the values of the parameters are taken as $r_1 = 3.8, r_2 = 0.8, r_3 = 0.2, a_{11} = 1.8, a_{12} = 0.1, a_{13} = 0.1, a_{21} = 0.6, a_{22} = 0.2, a_{23} = 0.5, a_{31} = 0.1, a_{32} = 1, a_{33} = 0.1, h = 45$, with initial points $(0.5, 0.5, 0.5)$

In discrete systems, one can find bifurcation and chaos. This phenomenon leads to the extinction of interacting species. Therefore, it is reasonable to check chaotic orbits. In this case, a hybrid control method is used to normalise the system. In Figure 2, we found chaotic behaviour of system (2) under a certain choice of system parameters, but this dynamic is absent for the continuous system (1) (Figure 5).

To our knowledge, there is no study related to discrete-time models that considers a predator-prey system with intraguild predation, focusing on Neimark-Sacker and period doubling bifurcations, chaos control, and global stability issues.

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Conflict of Interest Statement

The author declares that they have no conflict of interest.

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