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ON THE APPROXIMATION ALGORITHMS FOR SOLVING RICCATI DIFFERENTIAL EQUATIONS USING PIECEWISE CONSTANT ARGUMENTS METHOD

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ABSTRACT

This paper presents an efficient approach for determining an approximate solution to Riccati differential equations with variable coefficients. The approach is introduced as a differential equation with piecewise constant arguments, corresponding to the considered initial value problem, which depends on a positive integer . It is shown that this equation has a unique piecewise smooth solution, which serves as an approximate solution to the considered initial value problem for large . Numerical results are provided, demonstrating the efficiency and high accuracy of the proposed method.

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Introduction

This paper introduces a new method for obtaining an approximate solution to the following Riccati differential equation:

$$\begin{aligned} y' &= P(t)+Q(t)y+R(t)y^2, \quad t\in[0,1] \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where $P_0(\cdot)$, $Q(\cdot)$ and $R(\cdot)$ are real variable continuous functions on the interval . These functions play a significant role in many fields of applied science [1]. For example, the one-dimensional static Schrodinger equation is closely related to a Riccati differential equation. A solitary wave solution of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions, satisfying a projective Riccati equation [2]. Such problems also arise in the optimal control literature. Consequently, this problem has attracted the attention of many scholars. However, an analytical solution in explicit form seems unlikely to be found, except for certain special cases. For example, some Riccati equations with constant coefficients can be solved analytically using several methods [3]. As a result, numerical techniques or approximate approaches are typically required to obtain solutions. Adomian's decomposition method has been suggested for solving Riccati differential equations by El-Tawil *et al.* [4]. In subsequent studies [5-7], special Riccati differential equations were solved using He's variational iteration method (VIM), the homotopy perturbation method (HPM) and the iterated form of He's HPM. The accuracy of the solutions obtained was compared with those derived using Adomian's decomposition method. Geng [8-9] introduced the piecewise VIM for solving Riccati differential equations.

VIM, originally proposed by He [10-13], has been demonstrated by numerous authors to be a powerful mathematical tool for addressing a wide range of linear and non-linear problems [14-19].

In this paper, an approximate solution to Equation (1) is constructed. Firstly, a differential equation with piecewise constant arguments is introduced, which corresponds to the considered Riccati equation with initial conditions that depend on a positive integer N. The solution of the introduced differential equation is then defined as a piecewise smooth function with respect to N . It is shown that this equation has a unique piecewise smooth solution, which will serve as an approximate solution to the considered initial value problem for large N. A new calculation method is presented to find this approximate solution, reducing the complexity of the computational process. Numerical results are presented to demonstrate the effectiveness of the proposed method, and the errors between the approximate and exact solutions are estimated.

Differential Equations with Piecewise Constant Arguments

Let N be a positive integer. Consider the following differential equation with piecewise constant arguments of the form:

$$\begin{gathered} y'(t) = P(t) + Q(t)y(\frac{k-1}{N}) + R(t)y^2(\frac{k-1}{N}), \quad t \in [\frac{k-1}{N}, \frac{k}{N}) \\ y(0) = y_0, \quad k = 1, \dots, N \end{gathered} \tag{2}$$

A solution to Equation (2) is defined as follows:

Definition 1

A function $y(t) = y_N(t)$ is called a solution of the initial value problem in Equation (2) if the following conditions are satisfied:

(i) $y(t)$ is continuous on [0,1];

(ii) The derivative $y'(t)$ exists and is continuous on [0,1], with the possible exception at points $\frac{k}{N}, \quad k = 0,1,\dots,N$, where one-sided derivatives exist; and,

(iii) $y(t)$ satisfies the initial value problem Equation (2) on (0, 1) with the possible exception at points $\frac{k}{N}, \quad k = 0,1,\dots,N.$

Equation (2) is referred to as a differential equation with piecewise constant arguments, corresponding to the nonlinear differential Equation (1).

Theorem 1

For any positive integer , the initial value problem in Equation (2) has a unique solution y_N , as defined in Equation (5).

The following theorem claims that the function y_N can be an approximate solution for the initial value problem in Equation (1).

Theorem 2

For any $\varepsilon > 0$, there exists a positive number $N_0 = N(\varepsilon)$ such that for any positive integer number N with $N > N_0$, the inequality

$$\sup_{t\in[0,1]} | y_N'(t) - Q(t)y_N(t) - R(t)y_N^2(t) - P(t)| < \varepsilon \tag{3}$$

holds, where y_N is the solution of the initial value problem in Equation (2).

Proof of the Main Results

Proof of the Theorem

Let N be a positive integer number. For $t \in [0, \frac{1}{N})$ in Equation (2), the differential equation is:

$$y'(t) = P(t) + Q(t)y(0) + R(t)y^2(0).$$

Hence,

$$y(t) = \int_0^t (P(s) + Q(s)y(0) + R(s)y^2(0))ds + y(0), \quad t \in [0, \frac{1}{N}).$$

Since the functions $P(\cdot), Q(\cdot)$ and $R(\cdot)$ are continuous, the following limit exists:

$$y(\frac{1}{N}) = \lim_{t\to\frac{1}{N}-0} y(t). \tag{4}$$

Furthermore, we solve Equation (2) for $t \in [\frac{k-1}{N}, \frac{k}{N})$, where , is given by:

$$y(t) = \int_{\frac{k-1}{N}}^{t} (P(s) + Q(s)y(\frac{k-1}{N}) + R(s)y^2(\frac{k-1}{N}))ds + y(\frac{k-1}{N}), \quad t \in [\frac{k-1}{N}, \frac{k}{N}), \tag{5}$$

where

$$y(\frac{k-1}{N}) = \lim_{t\to\frac{k-1}{N}-0} y(t).$$

The function $y(t)$, as defined by Equation (5), is differentiable in $(0, \frac{1}{N}) \cup (\frac{1}{N}, \frac{2}{N}) \cup \cdots \cup (\frac{N-1}{N}, 1)$ and continuous on [0,1]. It follows that this function is the unique solution to Equation (2).

The following lemma claims that the set $\{y_N(\frac{k}{N})\}$ is uniformly bounded:

Lemma 1

There exists a positive number C such that for any N and $k \in \{1, \ldots, N\}$, the inequality

$$| y_N(\frac{k}{N}) | < C \tag{6}$$

holds.

Proof. Assume $| y_0 | \leq 1$. Otherwise, multiplying both sides of Equation (1) by $\frac{1}{y_0}$ leads to a similar equation with the initial condition $| y(0) | = 1$. The following notions are introduced:

$$P = \max_{0\le t\le 1} |P(t)|, \quad Q = \max_{0\le t\le 1} |Q(t)|, \quad R = \max_{0\le t\le 1} |R(t)|.$$

Let $y_k := y_N(\frac{k}{N})$. The set $\{y_k\}$ is partitioned as

$$\{y_0, y_1, \ldots, y_N\} = {}_N \cup {}_N,$$

where ${}_N = \{y_k : y_k \le 1\}$ and ${}_N = \{y_k : y_k > 1\}$. There exist numbers $k_i, n_i, s \in \{0, 1, \ldots, N\}$ such that the set M_N can be expressed as:

$$M_N = \{y_{k_1}, y_{k_1+1}, \ldots, y_{k_1+n_1}\} \cup \cdots \cup \{y_{k_s}, y_{k_s+1}, \ldots, y_{k_s+n_s}\}.$$

To prove Equation (6), it suffices to show the set $\{y_{k_i}, y_{k_i+1}, \ldots, y_{k_i+n_i}\}$ for $i = 1,\ldots,s$ is uniformly bounded. From Equation (5):

$$|y_k| = \left|\int_{\frac{k-1}{N}}^{\frac{k}{N}} (P(s) + Q(s)y(\frac{k-1}{N}) + R(s)y^2(\frac{k-1}{N}))ds + y(\frac{k-1}{N})\right|.$$

Therefore, the number y_{k_i} evaluates to

$$|y_{k_i}| \le (P + Q\,|y(\frac{k_i-1}{N})| + Ry^2(\frac{k_i-1}{N}))\frac{1}{N} + |y(\frac{k_i-1}{N})|$$

and hence

$$|y_{k_i}| \le (P+Q+R)\frac{1}{N} + 1, \tag{7}$$

since $|y_{k_i-1}| = |y(\frac{k_i-1}{N})| \le 1$.

For $y_k \in \{y_{k_i}, y_{k_i+1}, \ldots, y_{k_i+n_i}\}, i = 1, \ldots, s$, the evaluation of $|y_k|$ is as follows:

$$|y_k| \le (P + Q\,|y(\frac{k-1}{N})| + R\,|y(\frac{k-1}{N})|^2)\frac{1}{N} + |y(\frac{k-1}{N})| \le (\frac{P+Q+R}{N} + 1)y_{k-1}^2,$$

since $|y_{k-1}| = |y(\frac{k-1}{N})| > 1$. Thus,

$$|y_k| \le (\frac{P+Q+R}{N} + 1)y_{k-1}^2 \quad \text{for} \quad k = k_i + 1, \ldots, k_i + n_i.$$

By applying this recurrence inequality $k - k_i$ times, the following is obtained:

$$|y_k| \le (\frac{P+Q+R}{N} + 1)^{2(k-k_i)+1} y_{k_i}^{2(k-k_i)}.$$

From Equation (7), it follows that

$$|y_k| \le (\frac{P+Q+R}{N} + 1)^{2(k-k_i)+1} (\frac{P+Q+R}{N} + 1)^{2(k-k_i)}$$

or

$$|y_k| \le (\frac{P+Q+R}{N} + 1)^{4(k-k_i)+1}.$$

Hence,

$$|y_k| \le (\frac{P+Q+R}{N} + 1)^{4N+1}.$$

Since $\lim_{N\to\infty}(\frac{P+Q+R}{N}+1)^{4N} = exp(4(P+Q+R))$, according to the last inequality, $|y_k|$ is uniformly bounded for any N and $k \in \{k_i, k_i+1, \dots, k_i+n_i\}$ for $i = 1, \dots, s$. The inequality in Equation (6) is proven.

Proof of Theorem 2. For a solution $y_N(t)$, $t \in [\frac{k-1}{N}, \frac{k}{N}]$, of Equation (2), the following equality holds:

$$y_{N'}(t) - Q(t)y_N(t) - R(t)y_N^2(t) - P(t)$$
$$= -Q(t)(y_N(t) - y_N(\frac{k-1}{N})) - R(t)(y_N^2(t) - y_N^2(\frac{k-1}{N})) \quad (8)$$

Since

$$y_N(t) - y(\frac{k-1}{N}) = \int_{\frac{k-1}{N}}^{t} (P(s) + Q(s)y(\frac{k-1}{N}) + R(s)y^2(\frac{k-1}{N}))ds,$$

it follows that

$$|y_N(t) - y(\frac{k-1}{N})| \le (P + QC + RC^2)(t - \frac{k-1}{N})\frac{1}{N} \le (P + QC + RC^2)\frac{1}{N}.$$

Analogously, it can be shown that

$$|y_N(t) + y(\frac{k-1}{N})| \le (P + QC + RC^2)\frac{1}{N} + 2C.$$

Thus, Equation (8) becomes

$$|y'_N(t) - Q(t)y_N(t) - R(t)y_N^2(t) - P(t)| \le$$
$$(Q + R(P + QC + RC^2)\frac{1}{N} + 2C)(P + QC + RC^2)\frac{1}{N}. \quad (9)$$

Since the constants P Q, R and C do not depend of N, this inequality provides the proof of Theorem 2.

Numerical Results

To obtain an approximate solution of Equation (1) from Theorem 2, it is necessary to calculate y_N, which has the explicit form in Equation (5). When the functions $P(t), Q(t)$ and $R(t)$ have exact antiderivatives, the numerical calculation of y_N is straightforward.

A numerical method for calculating y_N is provided for the case in which the variable coefficients $P(t), Q(t)$ and $R(t)$ do not have exact antiderivatives. In this case, the explicit form of y_N in Equation (5), the functions $P(t), Q(t)$ and $R(t)$ can be replaced by the step functions $P(\frac{k-1}{N}), Q(\frac{k-1}{N})$ and $R(\frac{k-1}{N})$ for $t \in [\frac{k-1}{N}, \frac{k}{N})$. This method is based on the following assertion.

Lemma 2

If the function g is continuous on the closed interval $[a, b]$, then the sequence of piecewise continuous functions g_N defined on $[a, b]$ as $g_N(b) = g(b)$ and

$g_N(t) = g(\frac{k(b-a)}{N}),\ t \in [a + \frac{(k-1)(b-a)}{N},\ a + \frac{k(b-a)}{N}),\ k = 1,2,...,N$, uniformly converges to g on $[a,b]$.

Let

$$Y_N(t) = \int_{\frac{k-1}{N}}^{t} F(\frac{k-1}{N})ds - y(\frac{k-1}{N}), \quad t \in [\frac{k-1}{N}, \frac{k}{N}), \tag{10}$$

where

$$F(t) = P(t) + Q(t)y(t) + R(t)y^2(t)$$

$$Y_N(\frac{k-1}{N}) = \lim_{t \to \frac{k-1}{N} - 0} Y_N(t).$$

According to Lemma 2 and Theorem 2, we have the following theorem:

Theorem 3

For any $\varepsilon > 0$, there exists a positive integer $N_0 = N(\varepsilon)$ such that for any $N > N_0$, the estimation

$$\sup_{t, t \neq \frac{k}{N}} | Y'_N(t) - Q(t)Y_N(t) - R(t)Y_N^2(t) - P(t) | < \varepsilon$$

holds.

In order to verify the effectiveness of the suggested methods, several examples are considered, and the absolute errors between the approximate and exact solutions are provided.

Example 1

Let the Riccati equation with an initial value have the form:

$$y' = P(t) + t^2 y + \cos(t) y^2, \quad y(0) = 0,$$

where $P(t) = \frac{cos(\pi t)\pi}{1+t} - \frac{sin(\pi t)}{(1+t)^2} - \frac{t^2 sin(\pi t)}{1+t} - \frac{cos(t) sin(\pi t)^2}{(1+t)^2}$. Note that the function $P(t)$ is chosen so that the function $y(t) = \frac{sin(\pi t)}{1+t}$ is the exact solution to this problem. In this case, the graphs of the approximate solution y_N and the exact solution $y(t)$ to Equation (1) are compared in Figure 1.

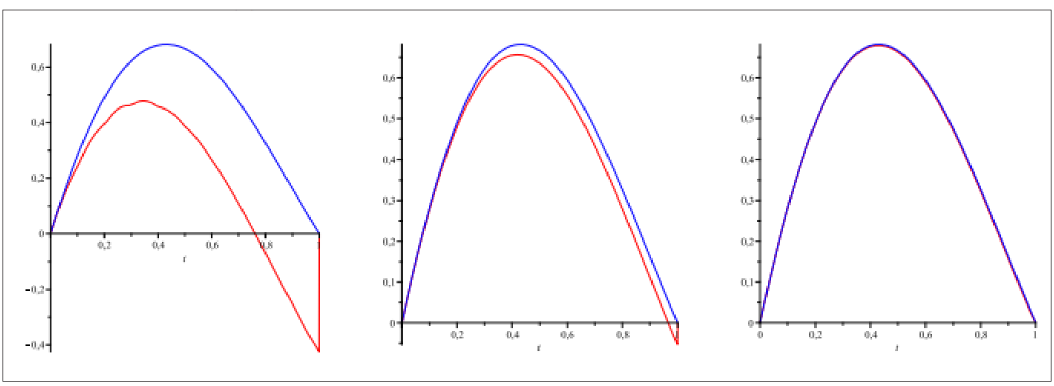

Figure 1: Graphs of the original (blue) and approximate (red) solutions for Example 1, with $N = 10$, $N = 100$, and $N = 1000$

Table 1: The absolute errors of the approximate and exact solutions for Example 1

N	$max \mid y(t) - y_N(t) \mid$
=10	0.425980525676032006
=100	0.0538799724869643422
=1000	0.00555050394969281836
=10000	0.000556747227769282508
=100000	0.0000558044840271352700

Example 2

Consider the Riccati equation with the initial value problem:

$$y' = P(t) + y + t^3 y^2, \quad y(0) = \frac{1}{2},$$

where $P(t) = -\frac{2sin(2\pi t)\pi}{1+e^t} - \frac{cos(2\pi t)e^t}{(1+e^t)^2} - \frac{cos(2\pi t)}{1+e^t} - \frac{t^3 cos(2\pi t)^2}{(1+e^t)^2}$. Note that the function $P(t)$ is chosen such that the function $y(t) = \frac{cos(2\pi t)}{1+e^t}$ is the exact solution for this problem. In this case, the graphs of the approximate solution y_N and the exact solution $y(t)$ to Equation (1) are compared in Figure 2.

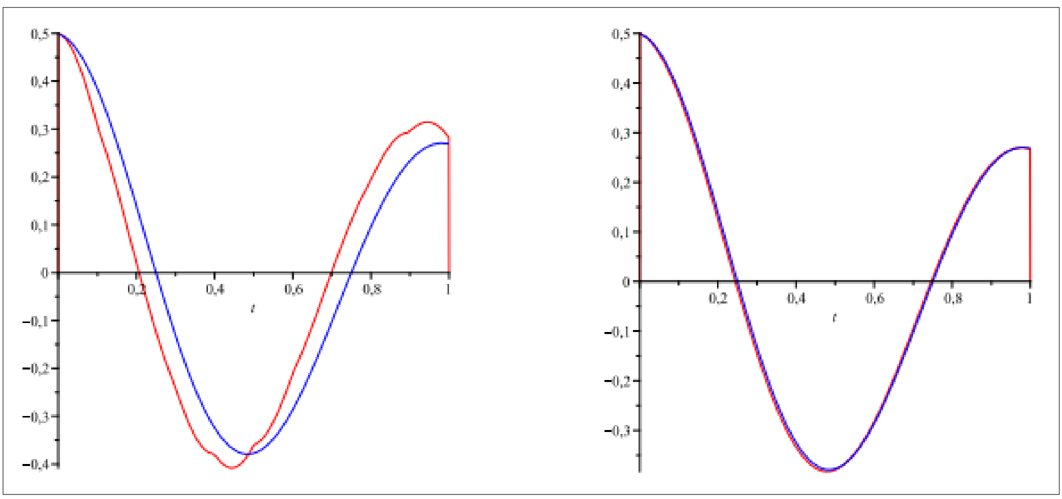

Figure 2: Graphs of the original (blue) and approximate (red) solutions for Example 2, with $N = 10$ and $N = 100$

Table 2: The absolute errors of the approximate and exact solutions for Example 2

N	$max \mid y(t) - y_N(t) \mid$
N=10	0.123854426043085331
N=100	0.0146512739668558674
N=1000	0.00148903739867298388
N=10000	0.000149134007816435854

Example 3

Consider the Riccati equation with the initial value problem:

$$y' = \frac{sin(t) - (2 - cos(t))^2}{2 - cos(t)} y + y^2, \quad y(0) = 1.$$

The exact solution to this problem is $y(t) = 2 - cos(t)$. In this case, the graphs of the approximate solution y_N and the exact solution $y(t)$ to Equation (1) are compared in Figure 3.

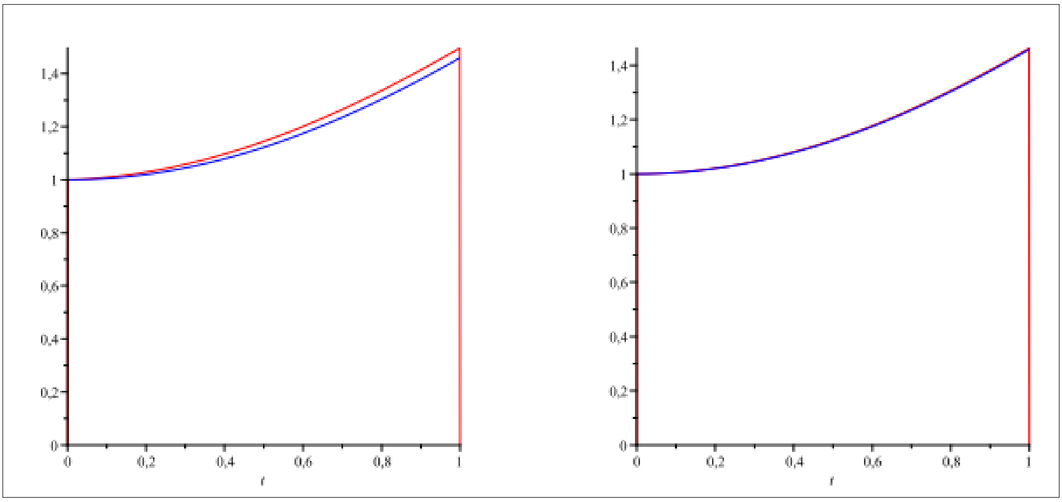

Figure 3: Graphs of the original (blue) and approximate (red) solutions for Example 3, with $N = 10$ and $N = 100$

Table 3: The absolute errors of the approximate and exact solutions for Example 3

N	$max\,\lvert y(t) - y_N(t)\rvert$
=10	0.0370616453557507386
=100	0.00414805584938660044
=1000	0.000420138739907593983
=10000	0.0000421275201351045681

Conclusions

This study presents a new method for obtaining approximate solutions to Riccati differential equations. First, a differential equation with piecewise constant arguments corresponding to the equation under consideration was constructed. The definition of the solution to the piecewise constant argument ordinary differential equations was introduced as a piecewise smooth function with respect to the positive integer N. It has been proven that this piecewise smooth solution approximates the solution to the considered initial value problem for the Riccati differential equation for large . Numerical results demonstrate the effectiveness of the proposed methods, and the errors between the approximate and the exact solutions were evaluated.

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Conflicts of Interest Statement

The authors declare no conflict of interest. The funders had no role in the design of the study, in the collection, analyses, or interpretation of data, in the writing of the manuscript, or in the decision to publish the results.

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