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ON THE SPECTRAL PROPERTIES OF THE MODEL OPERATOR ON A FERMIONIC FOCK SPACE

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ABSTRACT

This study examines a model operator $\mathbb{H}$ corresponding to a system of two identical fermions and another particle of a different nature. The operator acts on the direct sum of zero-, one-, and two-particle subspaces of the fermionic Fock space $\mathcal{F}_a(L^2(\mathbb{T}^d))$ over $L^2(\mathbb{T}^d)$, where $d \geq 3$. The essential spectrum of this operator is shown to consist of the union of at most four segments on the real axis. A formula for the corresponding resolvent operator is also explicitly derived.

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Introduction

Understanding the spectral properties of operators corresponding to systems of two, three or many particles has been of great importance in statistical physics [1, 2], solid-state physics [3, 4], and quantum field theory [5, 6]. For the important class of multi-particle systems, where the number of particles is not conserved, Sigal *et al.* [7] determined the location of the spectrum and proved the absence of singular continuous spectrum for Hamiltonians, employing geometric and commutator techniques. Additionally, accumulation points of the discrete spectrum were identified.

Conventionally, systems of particles interacting via pair potentials in a Fock space, where the number of particles is not conserved, can be reduced to several simpler subsystems. For example, instead of studying self-adjoint operators in the Fock space $\mathcal{H}^{(n)}$, one might consider the subspaces $\mathcal{H}^{(r)}$ corresponding to $r \leq n$ particles [1, 4-8]. Albeverio *et al.* [9, 10] studied model operators corresponding to two-boson systems and another particle of a different nature, thoroughly describing the essential spectrum. These model operators were described by so-called truncated operators, which correspond to subsystems of 0, 1 and 2 particles of the considered Fock space.

In this article, the essential spectrum of the model operator in the fermionic Fock space $\mathcal{F}_a\left(L^2(\mathbb{T}^d)\right)$ over $L^2(\mathbb{T}^d)$ is studied. The essential spectrum, which coincides with the spectrum of the Friedreich model $h(p)$, where $p \in \mathbb{T}^d$, is explicitly determined under certain smoothness conditions. It is shown that the essential spectrum of the operator $\mathbb{H}$ consists of a union of closed intervals on the real axis (Theorem 5.1). Furthermore, the explicit form of the corresponding resolvent operator is derived (Theorem 5.2).

The article is organised as follows: Section 1 is the introduction. Section 2 introduces the notations and describes the model operator. In Section 3, the channel operator and the Friedrichs models $H(p)$, where $p \in \mathbb{T}^d$, are defined, and their properties are discussed. Section 4 offers a brief review of the Faddeev-type integral equation for the operator $\mathbb{H}$. The main results and their proofs are presented in Section 5.

The Model Operator

The following notations are adopted: $\mathbb{T}^d = (-\pi, \pi]^d$ is the -dimensional torus, an abelian group $\mathbb{R}^d$ modulo $(2\pi\mathbb{Z})^d$. $L^2_{as}((\mathbb{T}^d)^2)$ denotes the subspace of antisymmetric functions of the Hilbert space $L^2((\mathbb{T}^d)^2)$. The following sets are defined:

$$\mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = L^2(\mathbb{T}^d), \text{ and } \mathcal{H}_2 = L^2_{as}((\mathbb{T}^d)^2).$$

Let I_j and $\langle \cdot,\cdot \rangle_j$, where j = 0,1,2, denote the identity operator and an inner product in $\mathcal{H}_j$, respectively. The direct sum of subspaces corresponding to 0, 1, and 2 particles in the Fock space $\mathcal{F}_a(L^2(\mathbb{T}^d))$ of fermions over $L^2(\mathbb{T}^d)$ is defined as:

$$\mathcal{H}^{(3)} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$$

For $g \in L^2(\mathbb{T}^d)$, the operators $\boldsymbol{g}: L^2(\mathbb{T}^d) \to \mathbb{C}$ and $\boldsymbol{g}^*: \mathbb{C} \to L^2(\mathbb{T}^d)$ are defined as:

$$\boldsymbol{g}(f) = \langle f, g \rangle_1, f \in L^2(\mathbb{T}^d),$$

$$\boldsymbol{g}^*(c) = c(\cdot)g, \ c \in \mathbb{C}$$

and the operators $L_g, L_{sg}: L^2((\mathbb{T}^d)^2) \to L^2(\mathbb{T}^d)$ and $L^*_g, L^*_{sg}: L^2(\mathbb{T}^d) \to L^2((\mathbb{T}^d)^2)$ are defined as:

$$L_g = I_1 \otimes \boldsymbol{g}, \quad L_{sg} = \boldsymbol{g} \otimes I_1,$$

$$L^*_g = I_1 \otimes \boldsymbol{g}^*, \quad L^*_{sg} = \boldsymbol{g}^* \otimes I_1.$$

For $i, j = 0,1,2$, let H_{ij}, where $i < j$ (resp. $i > j$) denote the annihilation (resp. creation) operator [5] in a Fock space. Annihilation operators reduce the number of particles by one in any state, while creation operators increase the number by one. For simplicity, it is assumed that the number of annihilation and creation operators is equal to one, i.e., $H_{ij} \equiv 0$ for all $|i - j| > 1$. The following operators are defined:

$$H_{01} = \boldsymbol{a}, H_{10} = \boldsymbol{a}^*, H_{11} = H^0_{11} + W,$$

$$H_{12} = L_b, H_{21} = \frac{1}{2}(L^*_b - L^*_{sb}), H_{22} = H^0_{22} - L^*_\varphi L_\varphi - L^*_{s\varphi} L_{s\varphi},$$

with H_{00}, H^0_{11}, and H^0_{22} being multiplication operators by the functions u_0, $u(\cdot)$, and $E(\cdot,\cdot)$ in $\mathcal{H}_0, \mathcal{H}_1$, and $\mathcal{H}_2$, respectively. The integral operator $W: \mathcal{H}_1 \to \mathcal{H}_1$ of a kernel $w(\cdot,\cdot)$ is expressed as:

$$Wf_1(p) = \int_{\mathbb{T}^d} w(p,t) f_1(t) dt$$

Here, u_0 is a constant number and $a(\cdot), b(\cdot), u(\cdot), \varphi(\cdot)$ are real functions continuous on $\mathbb{T}^d$, $E(\cdot,\cdot)$ is a real symmetric function continuous on $(\mathbb{T}^d)^2$, and $w(\cdot,\cdot) \in L^2((\mathbb{T}^d)^2)$ is a function that satisfies the self-adjointness property, i.,e., $w(p,q) = \overline{w(q,p)}, p, q \in \mathbb{T}^d$.

With these definitions, the truncated operator T is described by the matrix operator $\mathbb{H}$ in the space as:

$$\mathbb{H} = \begin{pmatrix} H_{00} & H_{01} & 0 \\ H_{10} & H_{11} & H_{12} \\ 0 & H_{21} & H_{22} \end{pmatrix} \tag{2.1}$$

This operator is a bounded, self-adjoint operator.

The Channel Operator and the Friedrichs Model

Next, a self-adjoint operator H_{ch} is considered, which acts in $\widehat{\mathcal{H}} = L^2(\mathbb{T}^d) \oplus L^2((\mathbb{T}^d)^2)$ as a matrix operator:

$$H_{ch} = \begin{pmatrix} H_{11} & \frac{1}{\sqrt{2}} L_b \\ \frac{1}{\sqrt{2}} L_b^* & H_{22}^0 - L_\varphi^* L_\varphi \end{pmatrix}.$$

This operator is called a channel operator associated with $\mathbb{H}$ [11]. Since H_{ch} is of a simpler form than $\mathbb{H}$, analysing its spectrum becomes considerably more straightforward. The relationship between the spectra of the channel operators and the operator $\mathbb{H}$ will be established later.

Let U_α be a multiplication operator by the function $\alpha(\cdot)$, defined as:

$$U_\alpha \begin{pmatrix} f_1(p) \\ f_2(p,q) \end{pmatrix} = \begin{pmatrix} \alpha(p) f_1(p) \\ \alpha(p) f_2(p,q) \end{pmatrix}, \quad \alpha \in L^2(\mathbb{T}^d).$$

It follows that H_{ch} commutes with U_α, i.e., $H_{ch} U_\alpha = U_\alpha H_{ch}$. This, along with the decomposition $\widehat{\mathcal{H}} = \int_{p\in\mathbb{T}^d} \mathcal{H}^{(2)} dp$, where $\mathcal{H}^{(2)} = \mathcal{H}_0 \oplus \mathcal{H}_1$, implies that the operator H_{ch} can also be written as an integral decomposition [12, Theorem XIII.84]:

$$H_{ch} = \int_{\mathbb{T}^d} \oplus \, H(p) dp, \tag{3.1}$$

Here, $H(p)$, $p \in \mathbb{T}^d$, being a Friedrichs model, is a bounded and self-adjoint operator defined in the space $\mathcal{H}^{(2)}$ as:

$$H(p) = H_0(p) + V$$

where $H_0(p)$ and V are operators acting as

$$H_0(p) = \begin{pmatrix} 0 & 0 \\ 0 & h_0(p) \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} u(p) & \frac{1}{\sqrt{2}} \boldsymbol{b} \\ \frac{1}{\sqrt{2}} \boldsymbol{b}^* & -\varphi^* \varphi \end{pmatrix}$$

respectively, where $H(p)$, $p \in \mathbb{T}^d$, is a multiplication operator by the function $e_p(\cdot) := E(p,\cdot)$, i.e.,

$$(h_0(p) f_1)(q) = e_p(q) f_1(q), \quad f_1 \in \mathcal{H}_1$$

Friedrichs models of this form were studied in [9, 10].

Due to the spectral properties of a decomposable operator [12, Theorem XIII.85], Equation (3.1) yields the following.

Theorem 3.1. For the spectrum of the operator , the following relation holds:

$$\sigma(H_{ch}) = \cup_{p\in\mathbb{T}^d} \{\sigma_d(H(p))\} \cup [E_{min}, E_{max}]$$

where is the discrete spectrum of , and:

$$E_{\min} = \min_{p,q\in\mathbb{T}^d} E(p,q), \quad E_{\max} = \max_{p,q\in\mathbb{T}^d}$$

3.1 Subsection. The spectral properties of the Friedrichs model $H(p)$, $p \in \mathbb{T}^d$.

As V is an operator of finite rank, the essential spectrum of coincides with the spectrum of $H_0(p)$, i.e.:

$$\sigma_{ess}(H(p)) = \sigma_{ess}(h_0(p)) = [m(p), M(p)], \quad p \in \mathbb{T}^d,$$

where

$$m(p) = \min_{q\in\mathbb{T}^d} e_p(q), \quad M(p) = \max_{q\in\mathbb{T}^d} e_p(q).$$

Next, for any $p \in \mathbb{T}^d$, the Fredholm determinant $\Delta(p,\cdot)$ of the operator $H(p)$ is defined as an analytic function in $\mathbb{C}\backslash[m(p), M(p)]$, given by:

$$\Delta(p; z) = (I_0 - \varphi r_0(p,z)\varphi^*)(H_{00}(p) - zI_0 - \tfrac{1}{2}\boldsymbol{b} r_0(p,z)\boldsymbol{b}^*)f_0 - \tfrac{1}{2}(\boldsymbol{b} r_0(p,z)\varphi^*)^2 f_0, \qquad (3.2)$$

where $f_0 = 1$. Using this equality and the forms of the included operators, the following expression is derived:

$$\Delta(p; z) = (1 - \int_{\mathbb{T}^d}\frac{\varphi^2(s)ds}{E(p,s)-z})(u(p) - z - \frac{1}{2}\int_{\mathbb{T}^d}\frac{b^2(s)ds}{E(p,s)-z}) - \frac{1}{2}(\int_{\mathbb{T}^d}\frac{b(s)\varphi(s)ds}{E(p,s)-z})^2,$$

where $r_0(p;z)$, $z \in \mathbb{C}\backslash[m(p), M(p)]$, is the resolvent of $h_0(p)$, $p \in \mathbb{T}^d$.

Lemma 3.2. For $p \in \mathbb{T}^d$, $z \in \mathbb{C}\backslash[m(p), M(p)]$, is an eigenvalue of the operator $H(p)$ *iff* $\Delta(p,z) = 0$.

Proof. Using the definition of the operator $H(p)$ from the equation:

$$H(p)f = zf, \quad f \in \mathcal{H}^{(2)}, \qquad (3.3)$$

the following system of equations is obtained:

$$\begin{cases} (u(p) - z)f_0 + \frac{1}{\sqrt{2}}\boldsymbol{b} f_1 = 0 \\ -\frac{1}{\sqrt{2}} r_0(p;z)\boldsymbol{b}^* f_0 + r_0(p;z)\varphi^*\varphi f_1 = f_1 \end{cases}, \; f = (f_0, f_1) \in \mathcal{H}^{(2)},$$

which is equivalent to:

$$\begin{cases} (H_{00}(p) - zI_0 - \frac{1}{2}\boldsymbol{b} r_0(p,z)\boldsymbol{b}^*)f_0 + \frac{1}{\sqrt{2}}\boldsymbol{b} r_0(p,z)\varphi^*\alpha = 0 \\ -\frac{1}{\sqrt{2}}\varphi r_0(p,z)\boldsymbol{b}^* f_0 + (I_0 - \varphi r_0(p,z)\varphi^*)\alpha = 0 \end{cases}, f_0, \alpha \in \mathbb{C}. \qquad (3.4)$$

Moreover, the solutions of Equations (3.3) and (3.4) are connected by the following relations:

$$f_0 = f_0, \alpha = \varphi f_1 \text{ and}$$

$$f_1(q) = r_0(p;z)(-\tfrac{1}{2}b(q)f_0 + \varphi(q)\alpha).$$

On the other hand, the determinant of Equation (3.4) is equal to $\Delta(p; z)$. Therefore, the equation $H(p)f = zf, f \in \mathcal{H}^{(2)}$, has a nontrivial solution if and only if $\Delta(p; z)$.

Let the notations $n_-(A, z)$, $z \leq inf\ \sigma_{\text{ess}}(A)$ (resp. $n_+(A, z)$, $z \geq sup\ \sigma_{\text{ess}}(A)$) represent the number of eigenvalues of $\mathbb{H}$ lying below z (resp. above z), counted with multiplicities.

Lemma 3.3. For any fixed $p \in \mathbb{T}^d$, the following hold:

(a) If $\varphi(\cdot)$ and are bounded (resp. unbounded), then

(b) $n_-(H(p), m(p)) \leq 1 \quad (resp.\ n_-(H(p), m(p)) \leq 2)$,

$n_+(H(p), M(p)) \leq 1$.

Proof. We can easily obtain that:

$$z_{1,2} = -\frac{u(p) \pm \sqrt{u^2(p) + 4||b||^2}}{2}, \qquad z_1 < 0 < z_2,$$

and these are simple eigenvalues of the matrix operator $V' = \begin{pmatrix} u(p) & \frac{1}{\sqrt{2}}\boldsymbol{b} \\ \frac{1}{\sqrt{2}}\boldsymbol{b}^* & 0 \end{pmatrix}$

According to the definition of the operator V, $\mathrm{Img}(V) = \mathbb{C} \oplus \langle b, \varphi \rangle$, where $\langle b, \varphi \rangle$ is the subspace spanned by b and φ. Using this and the facts that $-\varphi^*\varphi \leq 0$ and $\sigma_{ess}(V') = \{0\}$, the following are true for the operator V.

(i) If the functions $\varphi(\cdot)$ and $b(\cdot)$ are linearly bounded, then V has two positive eigenvalues (with multiplicities) and one (simple) negative eigenvalue; and,

(ii) If the functions $\varphi(\cdot)$ and $b(\cdot)$ are linearly unbounded, then V has only one (simple) positive and one (simple) negative eigenvalues.

According to the inequalities $H(p) \geq m(p) + V$, $H(p) \leq M(p) + V$, and the minimax principle, the numbers $n_-\big(H(p), m(p)\big)$ and $n_+\big(H(p), M(p)\big)$ satisfy the relations:

$$n_-\big(H(p), m(p)\big) \leq n_-\big(m(p) + V, m(p)\big) \text{ and } n_+(H(p), M(p)) \leq n_+\big(M(p) + V, M(p)\big),$$

respectively.

Using assertions (i) and (ii), as well as the relations $n_-\big(m(p) + V, m(p)\big) = n_-(V, 0)$ and $n_+\big(M(p) + V, M(p)\big) = n_+(V, 0)$, the proof is completed.

Corollary 3.4. Let $p \in \mathbb{T}^d$.

(a) If $\varphi(\cdot)$ and $b(\cdot)$ are linearly bounded (unbounded), the function $\Delta(p,\cdot)$ can have no more than one zero (two zeros) in the interval $\big(-\infty, m(p)\big)$.

(b) $\Delta(p,\cdot)$ may have only one zero in the interval $(M(p), \infty)$.

The Faddeev-type Equation

Denote the spectrum of the channel operator H_{ch} as:

$$\Sigma = \sigma(H_{ch}), \text{ then } \Sigma = [E_{min}, E_{max}] \cup \sigma_{\text{two}}$$

where $\sigma_{\text{two}} = \cup_{p \in \mathbb{T}^d} \sigma_d(H(p))$, i.e.,

$$\sigma_{\text{two}} = \{z \in \mathbb{R} \backslash [m(p), M(p)] : \Delta(p; z) = 0 \text{ for some } p \in \mathbb{T}^d\}$$

Theorem 4.1 The essential spectrum $\sigma_{ess}(\mathbb{H})$ of $\mathbb{H}$ coincides with the set Σ, *i.e.,*

$$\sigma_{ess}(\mathbb{H}) = \Sigma.$$

Proof. The proof can be found in [9].

4.1 Subsection. Faddeev-type integral equation

For any $a, b \in L^2(\mathbb{T}^d)$, the operators

$$L_{\boldsymbol{a}}R_{22}^0(z)L_b^*,\ L_{s\boldsymbol{a}}R_{22}^0(z)L_{sb}^*,\ L_{\boldsymbol{a}}R_{22}^0(z)L_{sb}^*,\ L_{s\boldsymbol{a}}R_{22}^0(z)L_b^*$$

are multiplication operators by the corresponding functions in $L^2(\mathbb{T}^d)$.

Furthermore, the operator

$$D(z) = I_1 - L_\varphi R_{22}^0(z)L_\varphi^* H_{11}^0 - zI_1 - \frac{1}{2}L_b R_{22}^0(z)L_b^* - \frac{1}{2}L_\varphi R_{22}^0(z)L_b^*,$$

is a multiplication operator by the function $\Delta(\cdot\,;z)$, $z \in \mathbb{C}\backslash[E_{min}, E_{max}]$, where $R_{22}^0(z) = (H_{22}^0 - zI_2)^{-1}$ is the resolvent of H_{22}^0.

Set $\overline{\mathcal{H}} = \overline{\mathcal{H}}_0 \oplus \overline{\mathcal{H}}_1 \oplus \overline{\mathcal{H}}_2$, where $\overline{\mathcal{H}}_0 = \mathcal{H}_0, \overline{\mathcal{H}}_1 = \mathcal{H}_1, \overline{\mathcal{H}}_2 = \mathcal{H}_1$.

For any $z \in \mathbb{C}\backslash\Sigma$, define the matrix operators *A*(*z*) and *K*(*z*), acting in the space $\overline{\mathcal{H}}$, as follows:

$$A(z) = \begin{pmatrix} A_{00}(z) & 0 & 0 \\ 0 & A_{11}(z) & A_{12}(z) \\ 0 & A_{21}(z) & A_{22}(z) \end{pmatrix} \quad \text{and} \quad K(z) = \begin{pmatrix} K_{00}(z) & K_{01}(z) & 0 \\ K_{10}(z) & K_{11}(z) & K_{12}(z) \\ 0 & K_{21}(z) & K_{22}(z) \end{pmatrix},$$

where $A_{ij}(z): \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i$, $i, j = 0, 1, 2$, are multiplication operators by functions $a_{ij}(p, z)$, defined as:

$$a_{00}(p,z) \equiv 1, a_{11}(p,z) = H_{11} - zI_1 - \frac{1}{2}L_b R_{22}^0(z)L_b^*, a_{12}(p,z) = \frac{1}{\sqrt{2}}L_b R_{22}^0(z)L_\varphi^*,$$
$$a_{21}(p,z) = \overline{a_{12}(p,z)} = \frac{1}{\sqrt{2}}L_\varphi R_{22}^0(z)L_b^*, a_{22}(p,z) = I_1 - L_\varphi R_{22}^0(z)L_\varphi^* \tag{4.1}$$

and the operators $K_{ij}(z): \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i$, $i, j = 0, 1, 2$, are defined by:

$$K_{00}(z) = (z+1)I_0 - H_{00}, \quad K_{01}(z) = -\boldsymbol{a}, \quad K_{10}(z) = K_{01}^*(z),$$
$$K_{11}(z) = -\frac{1}{2}L_b R_{22}^0(z)L_{sb}^* - W, \quad K_{12}(z) = \frac{1}{\sqrt{2}}L_b R_{22}^0(z)L_{s\varphi}^*,$$
$$K_{21}(z) = K_{12}^*(z), \quad K_{22}(z) = -L_\varphi R_{22}^0(z)L_{s\varphi}^*. \tag{4.2}$$

Note that for any $z \in \mathbb{C}\backslash\Sigma$, the operators $K_{ij}(z)$ are Hilbert-Schmidt operators, and so is *K*(*z*) in $\overline{\mathcal{H}}$.

Lemma 4.2. For any $z \in \mathbb{C}\backslash\Sigma$, $A(z)$, is a bounded invertable operator, and its inverse, $A^{-1}(z)$, is of the form:

$$A^{-1}(z) = \begin{pmatrix} B_{00}(z) & 0 & 0 \\ 0 & B_{11}(z) & B_{12}(z) \\ 0 & B_{21}(z) & B_{22}(z) \end{pmatrix},$$

where $B_{ij}(z): \overline{\mathcal{H}}_j \to \overline{\mathcal{H}}_i$, $i, j = 0,1,2$, are multiplication operators by the functions $b_{ij}(p, z)$, defined as:

$$b_{00}(p,z) \equiv 1, \quad b_{11}(p,z) = \frac{a_{22}(p,z)}{\Delta(p;z)}, \quad b_{12}(p,z) = -\frac{a_{12}(p,z)}{\Delta(p;z)},$$
$$b_{21}(p,z) = -\frac{a_{21}(p,z)}{\Delta(p;z)}, \quad b_{22}(p,z) = \frac{a_{11}(p,z)}{\Delta(p;z)}.$$

Proof. According to the definition, *A*(*z*) is a multiplication operator by the matrix

$$A(p,z) = \begin{pmatrix} a_{00}(p,z) & 0 & 0 \\ 0 & a_{11}(p,z) & a_{12}(p,z) \\ 0 & a_{12}(p,z) & a_{22}(p,z) \end{pmatrix}.$$

As a function of $p,\ A(p,z)$, is continuous in $\mathbb{T}^d$ and $det(A(p,z)) = \Delta(p,z) \neq 0,\ p \in \mathbb{T}^d$, for $z \notin \Sigma$. Therefore, it follows that $A^{-1}(z)$, the inverse of $A(z)$, is a multiplication operator by the matrix $A^{-1}(p,z)$ in $\overline{\mathcal{H}}$.

The following lemma establishes a connection between the eigenvalues of the operators $\mathbb{H}$ and $\hat{T}(z) = A^{-1}(z)K(z)$.

Lemma 4.3. $z \in \mathbb{C}\backslash\Sigma$ is an eigenvalue of $\mathbb{H}$ *iff* $\lambda = 1$ is an eigenvalue of $\hat{T}(z)$.

Proof. The proof of the lemma for $z \in (-\infty, E_{\min})$ can be found in [10]. For the right side of the essential spectrum, (E_{max}, ∞), it can be done analogously.

Formulation and Proof of the Main Results

The main results of the work are formulated below.

Theorem 5.1 The essential spectrum $\sigma_{ess}(\mathbb{H})$ of the operator $\mathbb{H}$ consists of a union of no more than four closed intervals.

Proof. Let $\sigma \subset \sigma(\mathbb{H})$ be the largest closed interval containing $[E_{min}, E_{max}]$, which may coincide with $\sigma = [E_{min}, E_{max}]$. Define:

$$\sigma_0 := \cup_{p\in\mathbb{T}^d} \{\sigma_d(H(p))\}\backslash\sigma. \tag{5.1}$$

If $\sigma_0 = \emptyset$, then $\sigma_{ess}(\mathbb{H})$ consists of only one closed interval σ.

Assume that $\sigma_0 \neq \emptyset$. Then, $\sigma_0 \cap \sigma = \emptyset$, and by the definition of the spectrum, the set σ_0 is closed. According to definiton (5.1), for any $p \in \mathbb{T}^d$, the operator $H(p)$ has an eigenvalue in σ_0.

Suppose that $[a,b] \subset \sigma_0$, where lie in the boundary set of $\sigma(\mathbb{H})$. Let G_ω be the set consisting of points $p \in \mathbb{T}^d$ such that $H(p)$ has an eigenvalue in $[a,b]$. It will be shown that $G_\omega = \mathbb{T}^d$. Let $p_0 \in G_\omega$. Then, by Theorem 3.1 and Lemma 4.2, there exists a number $z_0 \in [a,b]$ such that $\Delta(p_0, z_0) = 0$. However, for any $p \in \mathbb{T}^d$, $[a,b] \cap [E_{min}, E_{max}]$, and the function $\Delta(p,\cdot)$ is analytic in some region containing $[a,b]$. Consequently, either $\frac{\partial^1}{\partial z^1}\Delta(p_0,z_0) \neq 0$ or $\frac{\partial^2}{\partial z^2}\Delta(p_0,z_0) \neq 0$. By the implicit function theorem, there exist neighbourhoods $U(p_0) \subset \mathbb{T}^d$ and $U'(z_0) \subset [a,b]$ of the points p_0 and z_0, respectively, and a continuous function $z: U(p_0) \to U'(z_0)$ such that $\Delta(p, z(p)) \equiv 0$ for all $p \in U(p_0)$. By Lemma 4.2, the number $z(p) \in [a,b]$ is an eigenvalue of $H(p)$ for any $p \in U(p_0) \subset G_\omega$, which implies that G_ω is an open set.

Next, the closedness of the set G_ω is proven. Let $\{p_n\} \subset G_\omega$ be a sequence that converges to $p_0 \in \mathbb{T}^d$ and let $\{z(p_n)\} \subset [a,b]$ be an eigenvalue of the operator $H(p_n)$. Without loss of generality (or by choosing a subsequence), it can be assumed that:

$$\lim_{n\to\infty} z(p_n) = z_0 \in [a,b].$$

Since the function $\Delta(\cdot,\cdot)$ is continuous in $\mathbb{T}^d \times [a,b]$, it follows that:

$$0 \equiv \lim_{n\to\infty} \Delta(p_n, z(p_n)) = \Delta(p_0, z_0),$$

and, therefore, $p_0 \in G_\omega$, since $[a,b]$ is closed. Hence, the set G_ω is closed. As G_ω is both open and closed, it follows that $G_\omega = \mathbb{T}^d$.

According to Corollary 3.4, the discrete spectrum of $H(p), p \in \mathbb{T}^d$ contains no more than three eigenvalues. Therefore, the number of closed intervals $[a,b] \subset \sigma_0$ does not exceed three. Since

$\sigma_{ess}(H) = \sigma_0 \cup [E_{min}, E_{max}]$, it is concluded that the essential spectrum $\sigma_{ess}(\mathbb{H})$ consists of the union of no more than four segments.

5.1 Subsection. Resolvent operator of $\mathbb{H}$

Finally, the resolvent operator of $\mathbb{H}$ is considered in the form:

$$R(z) = \begin{pmatrix} R_{00}(z) & R_{01}(z) & R_{02}(z) \\ R_{10}(z) & R_{11}(z) & R_{12}(z) \\ R_{20}(z) & R_{21}(z) & R_{22}(z) \end{pmatrix}, \quad z \in \Sigma,$$

where $R_{ij}(z): \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0,1,2$, are its matrix entries.

The rows $\mathcal{R}_i(z): \mathcal{H}^{(d)} \to \mathcal{H}_i$, $i = 0,1,2$ are given by:

$$\mathcal{R}_i(z) = (R_{i0}(z), R_{i1}(z), R_{i2}(z)).$$

For $z \in \mathbb{C} \backslash [E_{min}, E_{max}]$, define:

$$\mathcal{R}_0(z) = \mathrm{diag}\{1, I_1, R_{22}^0(z)\}, H_0(z) = \mathrm{diag}\{I_0, I_1, H_{22}^0 - zI_2\},$$

where $R_{22}^0(z) = (H_{22}^0 - zI_2)^{-1}$ is the resolvent of the operator H_{22}^0.

Then, the following equation holds:

$$R(z) = \mathcal{R}_0(z) - \mathcal{R}_0(z)(H - zI - H_0(z))R(z),$$

i.e.,

$$\begin{pmatrix} \mathcal{R}_0(z) \\ \mathcal{R}_1(z) \\ \mathcal{R}_2(z) \end{pmatrix} = \mathcal{R}_0(z)I - \begin{pmatrix} (H_{00} - zI_0 - I_0)\mathcal{R}_0(z) + H_{01}\mathcal{R}_1(z) \\ H_{10}\mathcal{R}_0(z) + (H_{11} - zI_1 - I_1)\mathcal{R}_1(z) + H_{12}\mathcal{R}_2(z) \\ R_{22}^0(z)H_{21}\mathcal{R}_1(z) - R_{22}^0(z)(L_\varphi^* L_\varphi + L_{s\varphi}^* L_{s\varphi})\mathcal{R}_2(z) \end{pmatrix}. \tag{5.2}$$

Next, define a unitary operator $P: L_2^{as}((\mathbb{T}^d)^2) \to L_2^{as}((\mathbb{T}^d)^2)$ as:

$$Pf(p,q) = -f(q,p).$$

With this definition, the following relations hold:

$$L_{s\varphi}^* L_{s\varphi} = -PL_\varphi^* L_\varphi \quad \text{and} \quad L_{s\varphi}^* L_{s\varphi}\mathcal{R}_2(z) = -PL_\varphi^* L_\varphi \mathcal{R}_2(z).$$

Using these expressions in

$$\mathcal{R}_2(z) = R_{22}^0(z) - (R_{22}^0(z)H_{21}\mathcal{R}_1(z) - R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^* P)L_\varphi \mathcal{R}_2(z)),$$

Equation (5.2) simplifies to:

$$\begin{pmatrix} \mathcal{R}_0(z) \\ \mathcal{R}_1(z) \\ \mathcal{R}_2(z) \end{pmatrix} = \mathcal{R}_0(z) - Q_0(z) - H_1 X(z),$$

where

$$Q_0(z) = \begin{pmatrix} 0 \\ H_{12}R_{22}^0(z) \\ 0 \end{pmatrix}, \quad X(z) = \begin{pmatrix} \mathcal{R}_0(z) \\ \mathcal{R}_1(z) \\ L_\varphi \mathcal{R}_2(z) \end{pmatrix}$$

and

$$H_1 = \begin{pmatrix} H_{00} - (z+1)I_0 & H_{01} & 0 \\ H_{10} & H_{11} - (z+1)I_1 - H_{12}R_{22}^0(z)H_{21} & H_{12}R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) \\ 0 & R_{22}^0(z)H_{21} & -R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) \end{pmatrix}.$$

Denote $\hat{L} = \mathrm{diag}\{I_0, I_1, L_\varphi\}$. Then, $X(z) = (\mathcal{R}_0(z), \mathcal{R}_1(z), L_\varphi \mathcal{R}_2(z))^T$ can be rewritten as:

$$X(z) = \hat{L}\mathcal{R}_0(z) - \hat{L}Q_0(z) -$$
$$\begin{pmatrix} H_{00} - (z+1)I_0 & H_{01} & 0 \\ H_{10} & H_{11} - (z+1)I_1 - H_{12}R_{22}^0(z)H_{21} & H_{12}R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) \\ 0 & L_\varphi R_{22}^0(z)H_{21} & -L_\varphi R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) \end{pmatrix} X(z),$$

According to Equations (5.1) and (5.2), the following relations hold:

$$H_{00} - (z+1)I_0 = -I_0 - A_{00}(z) - B_{00}(z),$$
$$H_{11} - (z+1)I_1 - H_{12}(R_{22}^0(z), \quad H_{21} = -I_1 - A_{11}(z) - K_{11}(z),$$
$$-H_{12}R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) = A_{12}(z) - K_{12}(z)$$
$$L_\varphi R_{22}^0(z)H_{21} = A_{21}(z) - K_{21}(z)$$
$$L_\varphi R_{22}^0(z)(L_\varphi^* - L_{s\varphi}^*) = -I_2 + A_{22}(z) - K_{22}(z).$$

Therefore,

$$(A(z) - K(z))\boldsymbol{X}(z) = \hat{L}(\mathcal{R}_0(z) + Q_0(z)). \tag{5.3}$$

According to Lemma 4.3, if λ = 1 is not an eigenvalue of $\hat{T}(z)$, then Equation (5.3) has a solution:

$$\boldsymbol{X}(z) = (A(z) - K(z))^{-1}\hat{L}(\mathcal{R}_0(z) + Q_0(z))$$

The last equation and Equation (5.2) yield the following theorem.

Theorem 5.2. Let $z \in \mathbb{C} \backslash \sigma_{ess}(H)$. The resolvent of $\mathbb{H}$ is of the form:

$$R(z) = \mathcal{R}_0(z) - Q_0(z) - H_1(A(z) - K(z))^{-1}\hat{L}(\mathcal{R}_0(z) + Q_0(z)).$$

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Conflicts of Interest Statement

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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