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# COMPUTATION OF FIRST ZAGREB, SECOND ZAGREB AND FORGOTTEN INDICES OF GRAPHS BASED ON SPLITTING GRAPH

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**ABSTRACT**

The First Zagreb, Second Zagreb, and Forgotten indices are degree-based topological indices used to analyse the properties of chemical compounds. In graph theory, a molecular graph can represent the structural formula of a chemical compound. This paper aims to calculate the exact formulae of the First Zagreb, Second Zagreb, and Forgotten indices for the graphs obtained by applying operations on the splitting graph of a connected graph. Thus, a relation between the topological index of the original graph and the resultant graphs is established.

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## Introduction

Topological indices were originally developed for molecular graphs to study their chemical compounds' physicochemical, structural and other properties. Topological indices with their uses in Quantitative Structure-Property Relationship (QSPR) and Quantitative Structure-Activity Relationship (QSAR) are referred to in [1]. The degree-based structural descriptors have been a subject of detailed study since their induction from the first degree-based topological index named the First Zagreb Index in 1972 by I. Gutman, N. Trinajstić [2] while investigating the correlation value between the total $\pi$-electron energy and the structure of a molecule. Gutman *et al*. [3] developed their work and established the Second Zagreb index. In parallel, another index was introduced by Furtula and Gutman, which was, however, left untouched till 2015 [4]. Hence, the Forgotten index was renamed. Interestingly, it enhanced the physio-chemical applicability of the Zagreb index. This has improved the importance of more research on various topological indices, thereby reducing the number of experiments to be conducted, which is time-consuming and expensive. To understand some basic properties of First and Second Zagreb indices, one may read [5, 6]. Exact expressions for the Zagreb indices of some graph operations are presented in [7, 8, 9, 10].

Let $G$ be a graph with vertex set $V$ and edge set $E$. For an arbitrary vertex $u$ in $G$, $d_G(u)$ denotes the degree of the vertex $u$. The First Zagreb index, $M_1(G)$ of graph $G$ is the sum of squares of degrees of each vertex. It is also defined as the sum of the degrees of the end vertices of every edge of *G*. The Second Zagreb index, $M_2$ (*G*) of graph *G*, is defined as the sum of the product of the degrees of the end vertices of every edge of *G*. The Forgotten index, *F* (*G*) of graph *G*, is the sum of the cubes of degrees of each vertex.

$$M_1(G) = \sum_{u \in V} d_G(u)^2 = \sum_{uv \in E} d_G(u) + d_G(v)$$

$$M_2(G) = \sum_{uv \in E} d_G(u) d_G(v)$$

$$F(G) = \sum_{u \in V} d_G(u)^3$$

Simple, connected, undirected graphs will be considered throughout the paper. Graphs whose indices are formulated are defined below.

**Definition 1.1.** [11] Let *G* be a graph with vertex set $V = \{v_1, v_2, \dots, v_p\}$ and edge set *E*. Consider a set $V' = \{v_1', v_2', \dots, v_p'\}$ of $p$ isolated vertices. Make $v_i'$ adjacent to all the vertices in the open neighbourhood of $v_i$, for each $i = 1, 2, \dots, p$. The graph thus obtained is called the *splitting graph* of *G*, denoted by *splt*(*G*).

**Definition 1.2.** Let *G* be a graph with vertex set $V = \{v_1, v_2, \dots, v_p\}$ Let $splt(G)$ be the splitting graph of *G*. Take two copies of $splt(G)$ and let each $v_i \in V$ of one copy be adjacent to the corresponding vertex in the other copy. The resulting graph is called the *second splitting product* of *G*, denoted as $splt^{(2)}(G)$.

**Definition 1.3.** [12] The *splitting V-vertex join* graph, $G \veebar H$ is obtained from $splt(G)$ and *H* by joining every vertex of *H* with every vertex of *V*.

**Definition 1.4.** [12] The *splitting S-vertex join* graph, $G \barwedge H$ is obtained from $splt(G)$ and *H* by joining every vertex of *H* with every vertex of $V'$.

**Definition 1.5.** [13] The *splitting corona*, $G \boxminus H$ of *G* and *H* is obtained from $splt(G)$ and $p$ copies of *H* by joining every vertex of the $i^{th}$ copy of *H* to the $i^{th}$ vertex of $V'$, for $i = 1, 2, \dots, p$.

**Definition 1.6.** [13] The *splitting adds vertex corona*, $G \boxminus H$ of *G* and *H* is obtained from $splt(G)$ and $p$ copies of *H* by joining every vertex of the $i^{th}$ copy of *H* to the $i^{th}$ vertex of $V'$, for $i = 1, 2, \dots, p$.

**Definition 1.7.** [13] The *splitting neighbourhood corona*, $G \boxminus H$ of *G* and *H* is obtained from $splt(G)$ and $p$ copies of *H* by joining every vertex of the $i^{th}$ copy of *H* to the vertices adjacent to the $i^{th}$ vertex of $V'$, for $i = 1, 2, \dots, p$.

To avoid confusion in the proof of theorems in the subsequent sections, we shall denote the vertex of graph *F* obtained from the above definitions as *V* (*F*) and the degree of a vertex $u$ of *F* as $d_F(u)$.

## Main Results

### First Zagreb Index

The First Zagreb indices of the graphs defined above are calculated. Let the vertex set of *G* be $V = \{v_1, v_2, \dots, v_p\}$. Let the additional vertex set in the *splt*(*G*) be $V' = \{v_1', v_2', \dots, v_p'\}$. The *splt*(*G*) has vertex set $V \cup V'$. Let the vertex set of *H* be $U = \{u_1, u_2, \dots, u_q\}$.

**Theorem 2.1.** *For a connected graph G,*

$$M_1\big(splt(G)\big) = 5M_1(G).$$

*Proof.* Let the vertex set of *G* be $V = \{v_1, v_2, \dots, v_p\}$. Let the additional vertex set in the *splt*(*G*) be $V' = \{v_1', v_2', \dots, v_p'\}$. From Definition 1.1, the degree of $v_i$ in *splt*(*G*) is twice its degree in *G* and the degree of $v_i'$ in *splt*(*G*) is the degree of $v_i$ in *G*, i.e. $d_{splt(G)}(v_i) = 2d_G(v_i)$ and $d_{splt(G)}(v_i') = d_G(v_i)$.

$$\begin{aligned}
\therefore M_1\big(splt(G)\big) &= \sum_{u\in V(splt(G))} d_{splt(G)}(u)^2 \\
&= \sum_{v_i\in V} d_{splt(G)}(v_i)^2 + \sum_{v_i'\in V'} d_{splt(G)}(v_i')^2 \\
&= \sum (2d_G(v_i))^2 + \sum d_G(v_i)^2 \\
&= 5\sum d_G(v_i)^2 \\
&= 5M_1(G). \blacksquare
\end{aligned}$$

**Theorem 2.2.** *For a connected graph G with p vertices and $e_G$ edges,*

$$M_1\left(splt^{(2)}(G)\right) = 10M_1(G) + 2p + 16e_G.$$

*Proof.* From Definition 1.2, vertex $v_i$ of one copy of *splt*(*G*) is joined to its corresponding vertex in the other copy of *splt*(*G*). The degree of $v_i$ in $splt^{(2)}(G)$ is one more than twice its degree in *G*. The degree of $v_i'$ in $splt^{(2)}(G)$ is the degree of $v_i$ in *G*.

$$\begin{aligned}
\therefore M_1\left(splt^{(2)}(G)\right) &= 2\sum_{u\in V\left(splt^{(2)}(G)\right)} d_{splt^{(2)}(G)}(u)^2 \\
&= 2\left(\sum_{v_i\in V}(2d_G(v_i)+1)^2 + \sum_{v_i'\in V'} d_G(v_i)^2\right) \\
&= 10\sum_{v_i\in V} d_G(v_i)^2 + 2\sum_{v_i\in V} 1 + 8\sum_{v_i\in V} d_G(v_i).
\end{aligned}$$

Using the fundamental theorem of graph theory that states $\sum_{v_i\in V} d_G(v_i) = 2e_G$ and $\sum_{v_i\in V} d_G(v_i)^2 = M_1(G)$, the result follows.

**Theorem 2.3.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$M_1\big(G \veebar H\big) = 5M_1(G) + M_1(H) + pq(p+q) + 4pe_H + 8qe_G.$$

*Proof.* The *splt*(*G*) has vertex set $V \cup V'$ where $V = \{v_1, v_2, \dots, v_p\}$ is the vertex set of *G* and $V' = \{v_1', v_2', \dots, v_p'\}$ is the additional vertex set. Let the vertex set of *H* be $U = \{u_1, u_2, \dots, u_q\}$. From Definition 1.3, the degree of $v_i$ in $G \veebar H$ is the sum of twice its degree in *G* and *q*. The degree of $v_i'$ in $G \veebar H$ is the degree of $v_i$ in *G*. The degree of $u_j$ in $G \veebar H$ is *p* more than its degree in *H*.

$$\begin{aligned}\therefore M_1(G \veebar H) &= \sum_{u \in V(G \veebar H)} d_{G \veebar H}(u)^2 \\ &= \sum_{v_i \in V} (2d_G(v_i) + q)^2 + \sum_{v_i' \in V'} d_G(v_i)^2 + \sum_{u_j \in U} \left(d_H(u_j) + p\right)^2 \\ &= 4\sum_{v_i \in V} d_G(v_i)^2 + q^2 \sum_{v_i \in V} 1 + 4q \sum_{v_i \in V} d_G(v_i) + \sum_{v_i \in V} d_G(v_i)^2 \\ &\quad + \sum_{u_j \in U} d_H(u_j)^2 + p^2 \sum_{v_i \in V} 1 + 2p \sum_{u_j \in U} d_H(u_j).\end{aligned}$$

Using the fundamental theorem of graph theory, $\sum_{v_i \in V} d_G(v_i)^2 = M_1(G)$ and $\sum_{u_j \in U} d_H(u_j)^2 = M_1(H)$, the result follows.

**Theorem 2.4.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$\begin{aligned} M_1(G \barwedge H) &= 5M_1(G) + M_1(H) + pq(p+q) + 4pe_H + 4qe_G \\ &= M_1(G \veebar H) - 4qe_G. \end{aligned}$$

*Proof.* The proof is similar to that of Theorem 2.3. The degree of $v_i$ in $G \barwedge H$ is twice its degree in *G*. The degree of $v_i'$ in $G \barwedge H$ is the sum of $v_i$ in *G* and *q*. The degree of $u_j$ in $G \barwedge H$ is *p* more than its degree in *H*. The result thus follows.

**Theorem 2.5.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$M_1(G \boxminus H) = 5M_1(G) + pM_1(H) + pq^2 + 4pe_H + 8qe_G.$$

*Proof.* Let the vertex set of *G* be $V = \{v_1, v_2, \ldots, v_p\}$. The $splt(G)$ has vertex set $V \cup V'$, where $V' = \{v_1', v_2', \ldots, v_p'\}$ is the additional set of isolated vertices in $splt(G)$. Let the vertex set of *H* be $U = \{u_1, u_2, \ldots, u_q\}$. From Definition 1.5, the degree of $v_i$ in $G \boxminus H$ is the sum of twice its degree in *G* and *q*. The degree of $v_i'$ in $G \boxplus H$ is the degree of $v_i$ in *G*. The degree of $u_j$ in $G \boxminus H$ is one more than its degree in *H*. Note that there are *p* copies of *H* in $G \boxminus H$.

$$\begin{aligned}\therefore M_1(G \boxminus H) &= \sum_{u \in V(G \boxminus H)} d_{G \boxminus H}(u)^2 \\ &= \sum_{v_i \in V} (2d_G(v_i) + q)^2 + \sum_{v_i' \in V'} d_G(v_i)^2 + p\sum_{u_j \in U} \left(d_H(u_j) + 1\right)^2 \\ &= 4\sum_{v_i \in V} d_G(v_i)^2 + q^2 \sum_{v_i \in V} 1 + 4q \sum_{v_i \in V} d_G(v_i) + \sum_{v_i \in V} d_G(v_i)^2 \\ &\quad + p\sum_{u_j \in U} d_H(u_j)^2 + p \sum_{v_i \in V} 1 + 2p \sum_{u_j \in U} d_H(u_j).\end{aligned}$$

The result thus follows.

**Theorem 2.6.** *Let G and H be two connected graphs. Let G consist of p vertices and* $e_G$ *edges. Let H consist of q vertices and* $e_H$ *edges.*

$$\begin{aligned} M_1(G \boxplus H) &= 5M_1(G) + pM_1(H) + pq^2 + 4pe_H + 4qe_G \\ &= M_1(G \boxminus H) - 4qe_G. \end{aligned}$$

*Proof.* The proof is similar to that of Theorem 2.5. The degree of $v_i$ in $G \boxplus H$ is twice its degree in *G*. The degree of $v_i'$ in $G \boxplus H$ is the sum of $v_i$ in *G* and *q*. The degree of $u_j$ in $G \boxplus H$ is one more than its degree in *H*. Note that there are *p* copies of *H* in $G \boxplus H$. The result thus follows.

**Theorem 2.7.** *Let G and H be two connected graphs. Let G consist of p vertices and* $e_G$ *edges. Let H consist of q vertices and* $e_H$ *edges.*

$$\begin{aligned} M_1(G \boxtimes H) &= (q^2 + 5q + 5)M_1(G) + pM_1(H) + 8e_G e_H. \\ &= M_1(G \boxminus H) - 4qe_G. \end{aligned}$$

*Proof.* From Definition 1.7, the degree of $v_i$ in $G \boxtimes H$ is the sum of twice its degree in *G* and *q* times its degree in *G*. The degree of $v_i'$ in $G \boxtimes H$ is the degree of $v_i$ in *G*. Let $U_i = \{u_{ij} \mid j = 1, 2, \dots, q\}$ denote the vertex set of the $i^{th}$ copy of *H*. The degree of $u_{ij}$ in $G \boxtimes H$ is the sum of its degree in *H* and the degree of $u_i$ in *G*.

The sum of squares of the degrees of vertices in the $i^{th}$ copy of *H* is given by

$$\begin{aligned} &\sum_{u_{ij_i} \in U_i} \left(d(u_j) + d(v_i)\right)^2 \\ &= \sum_{u_j \in U} d(u_j)^2 + \sum_{u_j \in U} d(v_i)^2 + \sum_{u_j \in U} d(u_j) d(v_i) \\ &= M_1(H) + qd(v_i)^2 + 4d(v_i)e_H. \end{aligned}$$

Hence, the sum of squares of the degrees of vertices in all the *p* copies of *H* is given by $pM_1(H) + qM_1(G) + 8e_G e_H$

$$\begin{aligned} \therefore M_1(G \boxtimes H) &= \sum_{u \in V(G \boxtimes H)} d_{G \boxtimes H}(u)^2 \\ &= \sum_{v_i \in V} \left(2d(v_i) + qd(v_i)\right)^2 + \sum_{v_i' \in V'} d(v_i)^2 + \sum_i \sum_{u_{ij} \in U_i} \left(d(u_j) + d(v_i)\right)^2. \end{aligned}$$

The result thus follows.

## Second Zagreb Index

The second Zagreb indices of the graphs defined above are calculated.

**Theorem 2.8.** *Let G be a connected graph with p vertices.*

$$M_2(splt(G)) = 8M_2(G).$$

*Proof.* Let the vertex set of graph *G* with *p* vertices be $V = \{v_1, v_2, \dots, v_p\}$. Let the set of additional vertices in $splt(G)$ be $V' = \{v_1', v_2', \dots, v_p'\}$. The $splt(G)$ has vertex set $V \cup V'$. The edge set of

$splt(G)$ may be written as $E \cup E'$ where $E = \{v_i v_j | v_i, v_j \in V\}$ and $E' = \{v_i v_j' | v_i \in V, v_j' \in$ In $, V'$ for $i, j = 1, 2, \dots, p$ and $i \neq j\}$. In $splt(G)$, for every edge $v_1 v_j \in E$ there exist edges $v_i v_j', v_j v_i' \in E'$. By Definition 1.1, $splt(G)$ has $3e$ edges.

*Case 1*: For every $v_i v_j \in E$,

$$d_{splt(G)}(v_i) = 2d_G(v_i).$$

Hence,

$$\begin{aligned} &\sum_{v_i v_j \in E} d_{splt(G)}(v_i) d_{splt(G)}(v_j) \\ &= \sum_{v_i v_j \in E} 2d_G(v_i) 2d_G(v_j) \\ &= 4\mathrm{M}_2(\mathrm{G}). \end{aligned}$$

*Case 2*: For every $v_i v_j' \in E'$,

$$d_{splt(G)}(v_i) = 2d_G(v_i), \;\; d_{splt(G)}(v_j') = 2d_G(v_j).$$

Hence,

$$\begin{aligned} &\sum_{v_i v_j', v_j v_i' \in E'} d_{splt(G)}(v_i) d_{splt(G)}(v_j') + d_{splt(G)}(v_j) d_{splt(G)}(v_i') \\ &= \sum_{v_i v_j \in E} 2d_G(v_i) d_G(v_j) + 2d_G(v_j) d_G(v_i) \\ &= 4\mathrm{M}_2(\mathrm{G}). \end{aligned}$$

Combining both cases, $M_2(splt(G)) = 8M_2(G)$.

**Theorem 2.9.** *Let G be a connected graph with vertices and e edges.*

$$M_2\left(splt^{(2)}(G)\right) = 16M_2(G) + 10M_1(G) + 10e + p.$$

*Proof.* By Definition 1.2, $splt^2(G)$ has $6e + p$ edges. Let the vertices of one copy of $splt(G)$ be named as $v_1, v_2, \dots, v_p, v_1', v_2', \dots, v_p'$ and the vertices of the second copy be named as $u_1, u_2, \dots, u_p, u_1', u_2', \dots, u_p'$. The edge set of $splt^{(2)}(G)$ may be written as $E \cup E' \cup E''$ where

$$\begin{aligned} E &= \{v_i v_j, u_i u_j\}, \\ E' &= \{v_i v_j', u_i u_j'\} \text{ for } i, j = 1, 2, \dots, p \text{ and } i \neq j, \\ E'' &= \{v_i u_i\}. \end{aligned}$$

In the $splt^{(2)}(G)$, for every edge $v_i v_j \in E$ there exist edges $v_1 v_j', v_j v_i' \in E''$ and for every edge $u_i u_j \in E$ there exist edges $u_i u_j', u_j u_i' \in E'$.

*Case 1*: For every $v_i v_j, u_i u_j \in E$,

$$d_{splt^{(2)}(G)}(v_i) = 2d_{splt^{(2)}(G)}(v_i) = 2d_G(v_i) + 1.$$

Hence,

$$\sum_{v_iv_j\in E} d_{splt^{(2)}(G)}(v_i)d_{splt^{(2)}(G)}(v_j) + \sum_{u_iu_j\in E} d_{splt^{(2)}(G)}(u_i)d_{splt^{(2)}(G)}(u_j)$$
$$= 2\sum_{v_iv_j\in E}(2d_G(v_i)+1)(2d_G(v_j)+1)$$
$$= 2\left(4\sum_{v_iv_j\in E} d_G(v_i)d_G(v_j) + 2\sum_{v_iv_j\in E} d_G(v_i)+d_G(v_j) + \sum_{v_iv_j\in E} 1\right)$$
$$= 8M_2(G) + 4M_1(G) + 2e.$$

*Case 2*: For every $v_iv_j', u_iu_j' \in E'$,

$$d_{splt^{(2)}(G)}(v_i) = 2d_{splt^{(2)}(G)}(u_i) = 2d_G(v_i) + 1$$

and

$$d_{splt^{(2)}(G)}(v_j') = 2d_{splt^{(2)}(G)}(u_j') = 2d_G(v_j).$$

Hence,

$$\sum_{v_iv_j'\in E'} d_{splt^{(2)}(G)}(v_i)d_{splt^{(2)}(G)}(v_j') + \sum_{v_jv_i'\in E'} d_{splt^{(2)}(G)}(v_i)d_{splt^{(2)}(G)}(v_i')$$
$$+ \sum_{u_iu_j'\in E'} d_{splt^{(2)}(G)}(u_i)d_{splt^{(2)}(G)}(u_j') + \sum_{u_ju_i'\in E'} d_{splt^{(2)}(G)}(u_j)d_{splt^{(2)}(G)}(u_i')$$
$$= \sum_{v_iv_j\in E}\left((2d_G(v_i)+1)d_G(v_j) + (2d_G(v_j)+1)d_G(v_i)\right)$$
$$+ \sum_{v_iv_j\in E}\left((2d_G(v_i)+1)d_G(v_j) + (2d_G(v_j)+1)d_G(v_i)\right)$$
$$= 8M_2(G) + 2M_1(G).$$

*Case 3*: For every $v_iu_i \in E''$,

$$d_{splt^{(2)}(G)}(v_i) = 2d_{splt^{(2)}(G)}(u_i) = 2d_G(v_i) + 1.$$

Hence,

$$\sum_{v_iu_i\in E''} d_{splt^{(2)}(G)}(v_i)d_{splt^{(2)}(G)}(u_i)$$
$$= \sum_{v_i\in V}(2d_G(v_i)+1)^2$$
$$= 4M_1(G) + 8e + p.$$

Combining the three cases, $M_2\left(splt^{(2)}(G)\right) = 16M_2(G) + 10M_1(G) + 10e + p$.

**Theorem 2.10.** *Let G and H be two connected graphs. Let G consist of p vertices and* $e_G$ *edges. Let H consist of q vertices and* $e_H$ *edges.*

$$M_2(G \veebar H) = 8M_2(G) + 3qM_1(G) + M_2(H) + pM_1(H) + 2pq(2e_G + e_H) \\ +q^2e_G + p^2e_H + p^2q^2 + 8e_Ge_H.$$

*Proof.* By Definition 1.3, $G \veebar H$ has $3e_G + e_H + pq$ edges. Let the vertices of $G$ be named as $v_1, v_2, \ldots, v_p$ and the additional set of vertices in $splt(G)$ be named as $v'_1, v'_2, \ldots, v'_p$. Let the vertices of $H$ be named as $u_1, u_2, \ldots, u_q, u'_1, u'_2, \ldots, u'_p$. The edge set of $G \veebar H$ may be written as $E \cup E' \cup E'' \cup E'''$ where

$$\begin{aligned} E &= \{v_iv_j\}, \\ E' &= \{v_1v'_j, v_jv'_i\} \text{ for } i, j = 1, 2, \ldots, p \text{ and } i \neq j, \\ E'' &= \{u_ku_l\}, \\ E''' &= \{v_iu_k\} \text{ for } k, l = 1, 2, \ldots, q \text{ and } k \neq l. \end{aligned}$$

*Case 1*: For every $v_iv_j \in E$,

$$d_{G\veebar H}(v_i) = 2d_G(v_i) + q.$$

Hence,

$$\begin{aligned} &\sum_{v_iv_j\in E} d_{G\veebar H}(v_i)d_{G\veebar H}(v_j) \\ &= \sum_{v_iv_j\in E} (2d_G(v_i) + q)(2d_G(v_j) + q) \\ &= 4\sum_{v_iv_j\in E} d_G(v_i)d_G(v_j) + 2q\sum_{v_iv_j\in E} d_G(v_i) + d_G(v_j) + q^2\sum_{v_iv_j\in E} 1 \\ &= 4M_2(G) + 2qM_1(G) + q^2e_G. \end{aligned}$$

*Case 2*: For every $v_iv'_j, v_jv'_i \in E'$,

$$d_{G\veebar H}(v_i) = 2d_G(v_i) + q, \qquad d_{G\veebar H}(v'_j) = d_G(v_j).$$

Hence,

$$\begin{aligned} &\sum_{v_iv'_j\in E'} d_{G\veebar H}(v_i)d_{G\veebar H}(v'_j) + \sum_{v_jv'_i\in E'} d_{G\veebar H}(v_i)d_{G\veebar H}(v'_i) \\ &= \sum_{v_iv_j\in E} \Big((2d_G(v_i) + q)d_G(v_j) + (2d_G(v_j) + q)d_G(v_i)\Big) \\ &= 4M_2(G) + qM_1(G). \end{aligned}$$

*Case 3*: For every $u_ku_l \in E''$,

$$d_{G\veebar H}(u_k) = d_H(u_k) + p.$$

Hence,

$$\sum_{u_k u_l \in E''} d_{G\underline{\vee}H}(u_k) d_{G\underline{\vee}H}(u_l) = M_2(H) + pM_1(H) + p^2 e_H.$$

*Case 4*: For every $v_i u_k \in E'''$,

$$d_{G\underline{\vee}H}(v_i) = 2d_G(v_i) + q, \qquad d_{G\underline{\vee}H}(u_k) = 2d_H(u_k) + p.$$

$$\therefore d_{G\underline{\vee}H}(v_i) d_{G\underline{\vee}H}(u_k) = 2d_G(v_i)d_H(u_k) + qd_H(u_k) + 2pd_G(v_i) + pq.$$

By Definition 1.3, there exists an edge between every $v_i \in G$ and $u_k \in H$. For a fixed $u_k$ and varying $v_i$,

$$\sum d_{G\underline{\vee}H}(v_i) d_{G\underline{\vee}H}(u_k) = 4e_G d_H(u_k) + pqd_H(u_k) + 4pe_G + p^2 q.$$

Allowing $u_k$ to vary,

$$\sum_{v_i u_k \in E'''} d_{G\underline{\vee}H}(v_i) d_{G\underline{\vee}H}(u_k) = 8e_G e_H + 2pqe_H + 4pqe_G + p^2 q^2.$$

Combining all the cases, we attain the value of $M_2\left(G \underline{\vee} H\right)$.

**Theorem 2.11.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$M_2\left(G \overline{\wedge} H\right) = 8M_2(G) + 2qM_1(G) + M_2(H) + pM_1(H) + 2pq(e_G + e_H) \\ + p^2 e_H + p^2 q^2 + 4e_G e_H.$$

*Proof.* By Definition 1.4, $G \overline{\wedge} H$ has $3e_G + e_H + pq$ edges. Working on similar grounds as proved in Theorem 2.10, we get the result bearing in mind the following:

$$d_{G\overline{\wedge}H}(v_i) = 2d_G(v_i),$$

$$d_{G\overline{\wedge}H}(v_i') = d_G(v_i) + q,$$

$$d_{G\overline{\wedge}H}(u_k) = d_H(u_k) + p.$$

**Theorem 2.12.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$M_2(G \boxminus H) = 8M_2(G) + 3qM_1(G) + pM_2(H) + pM_1(H) + q^2 e_G \\ + pe_H + (4e_G + pq)(2e_H + q).$$

*Proof.* By Definition 1.5, $G \boxminus H$ has $3e_G + pe_H + pq$ edges. Let the vertices of $G$ be named as $v_1, v_2, \dots, v_p$ and the additional set of vertices in $splt(G)$ be named as $v_1', v_2', \dots, v_p'$. Let the vertices of the $i^{th}$ copy of $H$ be named as $u_{i1}, u_{i2}, \dots, u_{iq}$ where $i = 1,2, \dots, p$. If $\{v_i v_j | v_i \text{ adjacent to } v_j \text{ in } G\}$ denotes the edge set of $G$ and $\{u_k u_l | u_k \text{ adjacent to } u_l \text{ in } H\}$ denotes the edge set in $H$, the edge set of $G \boxminus H$ may be written as $E \cup E' \cup (\bigcup_i E_i'') \cup E'''$ where:

$$E = \{v_i v_j\},$$
$$E' = \{v_1 v_j', v_j v_i'\}$$
$$E'' = \{u_{ik} u_{il}\},$$
$$E''' = \{v_i u_{ik}\} \text{ for } i,j = 1,2,\dots,p, i \neq j \text{ and } k,l = 1,2,\dots,q, k \neq l.$$

*Case 1*: For every $v_i v_j \in E$,

$$d_{G \boxminus H}(v_i) = 2d_G(v_i) + q.$$

Hence,

$$\sum_{v_i v_j \in E} d_{G \boxminus H}(v_i) d_{G \boxminus H}(v_j) = 4M_2(G) + 2qM_1(G) + q^2 e_G.$$

*Case 2*: For every $v_i v_j', v_j v_i' \in E'$,

$$d_{G \boxminus H}(v_i) = 2d_G(v_i) + q, \qquad d_{G \boxminus H}(v_j') = d_G(v_j).$$

Hence,

$$\sum_{v_i v_j' \in E'} d_{G \boxminus H}(v_i) d_{G \boxminus H}(v_j') + \sum_{v_j v_i' \in E'} d_{G \boxminus H}(v_i) d_{G \boxminus H}(v_i')$$
$$= 4M_2(G) + qM_1(G).$$

*Case 3*: For every $u_{ik} u_{il} \in E_i''$,

$$d_{G \boxminus H}(u_k) = d_H(u_k) + 1.$$

Hence,

$$\sum_{u_{ik} u_{il} \in E_i''} d_{G \boxminus H}(u_{ik}) d_{G \boxminus H}(u_{il}) = M_2(H) + M_1(H) + e_H.$$

Considering the $p$ copies of $H$, $\sum_{\bigcup_i E_i''} d_{G \boxminus H}(u_{ik}) d_{G \boxminus H}(u_{il}) = p(M_2(H) + M_1(H) + e_H)$.

*Case 4*: For every $v_i u_{ik} \in E'''$,

$$d_{G \boxminus H}(v_i) d_{G \boxminus H}(u_k) = (2d_G(v_i) + q)(d_H(u_{ik}) + 1).$$

For a fixed $v_i$,

$$\sum d_{G \boxminus H}(v_i) d_{G \boxminus H}(u_{ik}) = (2d_G(v_i) + q)(2e_H + q).$$

Allowing $v_i$ to vary,

$$\sum_{v_i u_{ik} \in E'''} d_{G \boxminus H}(v_i) d_{G \boxminus H}(u_{ik}) = (4e_G + pq)(2e_H + q).$$

Combining all the cases, we attain the value of $M_2(G \boxminus H)$.

**Theorem 2.13.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

$$M_2(G \boxplus H) = 8M_2(G) + 2qM_1(G) + pM_2(H) + pM_1(H) + pe_H + (2e_G + pq)(2e_H + q).$$

*Proof.* By Definition 1.6, $G \boxtimes H$ has $3e_G + pe_H + pq$ edges. Working on similar grounds as proved in Theorem 2.12, we get the result bearing in mind the following:

$$d_{G\boxplus H}(v_i) = 2d_G(v_i),$$

$$d_{G\boxplus H}(v_i') = d_G(v_i) + q,$$

$$d_{G\boxplus H}(u_{ik}) = d_H(u_k) + 1.$$

**Theorem 2.14.** *Let G and H be two connected graphs. Let G consist of p vertices and* $e_G$ *edges. Let H consist of q vertices and* $e_H$ *edges.*

$$M_2(G \boxtimes H) = (q+2)(3q+4)M_2(G) + (2q+5)e_H M_1(G) + pM_2(H) + 2e_G M_1(H).$$

*Proof*. By Definition 1.7, $G \boxtimes H$ has $3e_G + pe_H + 2qe_G$ edges. Working on similar grounds as in previous theorems, cases 1 and 2 are obtained, bearing in mind that:

$$d_{G\boxtimes H}(v_i) = 2(q+2)d_G(v_i),$$

$$d_{G\boxtimes H}(v_i') = d_G(v_i).$$

Let the vertices of the $i^{th}$ copy of *H* be named as $u_{i1}, u_{i2}, \dots, u_{iq}$ where $i = 1,2,\dots,p$. Let $\{v_i v_j | v_i \text{ adjacent to } v_j \text{ in } G\}$ denotes the edge set of *G* and $\{u_k u_l | u_k \text{ adjacent to } u_l \text{ in } H\}$ denotes the edge set in *H*.

*Case 3*: For every $u_{ik}u_{il}$ in the $i^{th}$ copy of *H*,

$$d_{G\boxtimes H}(u_{ik}) = d_H(u_k) + d_G(v_i).$$

Hence,

$$\sum_{u_{ik}u_{il}} d_{G\boxtimes H}(u_{ik}) d_{G\boxtimes H}(u_{il}) = M_2(H) + d_G(v_i)M_1(H) + e_H d_G^2(v_i).$$

Considering the *p* copies of *H*, $\sum d_{G\boxtimes H}(u_{ik}) d_{G\boxtimes H}(u_{il}) = pM_2(H) + 2e_G M_1(H) + e_H M_1(G)$.

*Case 4*: For every edge $v_i u_{jk}$, whenever $v_i$ and $v_j$ are adjacent in *G*,

$$d_{G\boxtimes H}(v_i) d_{G\boxtimes H}(u_{jk}) = (q+2)d_G(v_i)\left(d_H(u_k) + d_G(v_j)\right).$$

Adding and reorganising the terms,

$$\begin{aligned}
&\sum d_{G\boxtimes H}(v_i) d_{G\boxtimes H}(u_{jk}) \\
&= \sum_{v_i v_j} (q+2)d_G(v_i)\left(d_H(u_k) + d_G(v_j)\right) \\
&= 2e_H(q+2)\sum_{v_i v_j} d_G(v_i) + d_G(v_j) + 2q(q+2)\sum_{v_i v_j} d_G(v_i)d_G(v_j) \\
&= 2e_H(q+2)M_1(G) + 2q(q+2)M_2(G).
\end{aligned}$$

Combining all the cases, we attain the value of $M_2(G \boxtimes H)$.

**Forgotten Index**

The Forgotten index of the graphs defined above is calculated.

**Theorem 2.15.** *Let G and H be two connected graphs. Let G consist of p vertices and $e_G$ edges. Let H consist of q vertices and $e_H$ edges.*

i. $F(splt(G)) = 9F(G)$;

ii. $F\left(splt^{(2)}(G)\right) = 18F(G) + 24M_1(G) + 2p + 24e_G$;

iii. $F(G \veebar H) = 9F(G) + F(H) + 12qM_1(G) + 3pM_1(H) + pq^3 + p^3q + 12q^2e_G + 6p^2e_H$;

iv. $F(G \barwedge H) = 9F(G) + F(H) + 3qM_1(G) + 3pM_1(H) + pq^3 + p^3q + 6q^2e_G + 6p^2e_H$
$= F(G \veebar H) - 9qM_1(G) - 6q^2e_G$;

v. $F(G \boxminus H) = 9F(G) + pF(H) + 12qM_1(G) + 3pM_1(H) + pq^3 + pq + 12q^2e_G + 6p^2e_H$;

vi. $F(G \boxplus H) = 9F(G) + pF(H) + 3qM_1(G) + 3pM_1(H) + pq^3 + pq + 6q^2e_G + 6p^2e_H$
$= F(G \boxminus H) - 9qM_1(G) - 6q^2e_G$;

vii. $F(G \boxtimes H) = ((q+2)^3 + q + 1)F(G) + pF(H) + 6e_HM_1(G) + 6e_GM_1(H)$.

*Proof.* Since $F(G) = \sum_{u \in V} d_G(u)^3$, the proof is similar to that of the theorems proved in section 2.1.

### Conclusions

The First Zagreb index, the Second Zagreb index and the Forgotten index of splitting graph $splt(G)$, second splitting product graph $splt^{(2)}(G)$, splitting *V*-vertex join graph $G \veebar H$, splitting *S*-vertex join graph $G \barwedge H$, splitting corona graph $G \boxminus H$, splitting add vertex corona graph $G \boxplus H$, splitting neighbourhood corona graph $G \boxtimes H$, are calculated. The study shows that the indices of the resultant graphs depend solely on the number of vertices and edges and the indices of the original graph.

### Conflict of Interest Statement

The authors declare that they have no conflict of interest.

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