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LOCAL DERIVATION OF REAL OPERATOR ALGEBRAS

KARIMOV U.S.*

National University of Uzbekistan, 100174 Toshkent, Uzbekistan.

**Corresponding author: karimovulugbe07@gmail.com*

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ABSTRACT

Local derivations of real W*-algebras are considered. Since in the Hermitian part of the real W*-algebra a local derivation is studied in the same way as in the complex case, here we consider the Skew-Hermitian part of the real W*-algebra. In the case of factors, the final result has been obtained, namely, it has been proven that any local derivation of real (complex) factors is an inner derivation.

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Introduction

In [1], the author considered local derivations and proved that local derivations on the von Neumann algebra are ordinary derivations. In [2], the author generalized the result of [1] for arbitrary C*-algebras. The idea of proving them is very simple, namely, for local differentiation δ it was enough to show the fulfillment of the Leibniz equality for projections of the algebra. Since the lattice of algebra projections generates the Hermitian part of the algebra, then for δ the Leibniz equality also holds for Hermitian elements of the algebra. On the other hand, the Hermitian part of the algebra generates the algebra itself, namely, every element of algebra is a linear combination of two Hermitian elements, then δ satisfies the Leibniz condition for any element of the algebra, therefore it is a derivation. In the real case, i.e. for real C*-algebras this is not enough, i.e. considering problems on the Hermitian part of algebra is not enough. Because in the real case the skew–adjoint part does not contain a single projection. Therefore, in this article we will mainly consider local derivation in the skew-adjoint part of the algebra. Recall that in [3] a real analogue of Kadison's result was obtained.

Preliminaries

Given an algebra, a map $\delta : R \rightarrow R$ is called a *derivation*, if it is linear and , $\delta(xy)= \delta(x)y+ x\delta(y)$, $\forall x, y \in R$. An operator $\delta : R \rightarrow R$ is called a *local derivation*, if it is linear and for every $x \in R$, there is a derivation $\delta_x : R \rightarrow R$ with $\delta(x)= \delta_x(x)$.

If H is a complex Hilbert space, then by $B(H)$ we denote the algebra of all linear bounded operators on space H. The weak topology on $B(H)$ is the topology, generated by seminorms of the form: $\rho(a)=|(\xi,a\eta)|,\ a\in B(H),\ \xi,\eta\in H$. A weakly closed *-subalgebra containing the identity operator **1** is called a W*-algebra. W*-algebra is also called von Neumann algebra.

A real Banach *-algebra R is called a real C*-algebra, if $||a^*a||=||a||^2$ and an element $1+a^*a$ is invertible for any $a\in R$. A real C*-algebra R for which $R+iR$ is a complex von Neumann algebra (W*-algebra) is called a real von Neumann algebra (real W*-algebra). E. Stormer gave another equivalent definition of a real W*-algebra: real *-algebra R in $B(H)$ with **1** called a real W*-algebra, if it is a weakly closed and $R\cap iR=\{0\}$ ([5],[6]).

Main Results

As stated above, R. Kadison in [1] proves that any local derivation of a W*-algebra into the dual Banach M-bimodule, which is continuous, is a derivation, and therefore is an inner derivation. In [2] B. Johnson generalised Kadison's results for C*-algebras and proved that if A is a C*-algebra, then any continuous local derivation of A into the Banach A-bimodule is continuous and hence it is a derivation. It was enough for them to show that the Leibniz equality holds for projections of the algebra.

The results of R. Kadison and B. Johnson can be easily transferred to the real case, i.e., for the self-adjoint elements of a real W*- (or C*)-algebra, the Leibniz identity holds:

Theorem 1 ([3]). *Let A be a real W*-algebra, $\delta: A\to A$ be a norm-continuous local derivation. Then, for any projections f and e :*

$$\delta(ef)=\delta(e)f+e\delta(f),$$

therefore, for any hermitian elements a and b we obtain $\delta(ab)=\delta(a)b+a\delta(b)$.

But, as stated above, the hermitian part of the real C*- and W*-algebras does not generate the algebra itself. There is also the skew-hermitian (skew-adjoint) part of the algebra. Exactly, any element of algebra is the sum of two elements, one of which is a hermitian element, and the other is a skew-hermitian element:

$$x=\frac{x+x^*}{2}+\frac{x-x^*}{2}.$$

Before considering the skew-hermitian part, let's consider the case factor. Recall that the ring A is called

- *semiprime*, if from $aAa=\{0\}$ it follows $a=0$;
- *prime*, if from $aAb=\{0\}$ it follows $a=0$ or $b=0$.

It is known that a real or complex W*-algebra is prime only if it is a factor. Let us present one auxiliary result from [6].

Theorem 2 [6]. *If A is a prime ring and A contains a nontrivial idempotent, then any local derivation of A is a derivation.* Since a real (or complex) W*-algebra always contains a nontrivial idempotent, then we have the following result.

Theorem 3. *If A is a real or complex factor, then every local derivation of A is a derivation, consequently it is an inner derivation.*

Now, let A be a real *-algebra and let $A_k=\{a\in A: a^*=-a\}$ is the skew-hermitian part. It is known that A_k is a (real) Lie algebra with respect to the brackets $[a,b] = ab - ba$. We want to move from ordinary (local) derivation to Lie (local) derivation $D([a,b]) = Da,b] + [a,Db]$ and, using the corresponding results for them, return again to ordinary derivation.

Recall that derivation of the Lie algebra L is a linear mapping $D : L \to L$, which satisfies Leibniz's identity, i.e. $D([a,b]) = [Da,b] + [a,Db]$, for all $a,b \in L$. In fact, this definition can be given for any algebra: a Lie-derivation or *L-derivation* of an (associative) algebra A is a linear mapping $D : A \to A$ which satisfies Leibniz's identity.

Theorem 4. *Let A be an (associative) algebra. Then the following statements are true*

I. *any derivation $D : A \to A$ is a L-derivation;*

II. *any local derivation $\delta : A \to A$ is a local L-derivation, i.e. at each point is a L-derivation;*

III. *for inner L-derivation (i.e. L-derivation of the form: $D_a(x) = [a,x]$) the converse is also true. That is inner (local) L-derivation is (local) derivation.*

Proof. I. If $D : A \to A$ is a derivation, then for any $a,b \in A$ we have

$$D([a,b]) = D(ab - ba) = Da \cdot b + a \cdot Db - Db \cdot a - b \cdot Da$$

On the other hand, we have $[Da,b]) = Da \cdot b - b \cdot Da$ and $[a,Db]) = a \cdot Db - Db \cdot a$. Hence, we obtain $[Da,b]) + [a ,Db] = D ([a,b])$, i.e. D is a L-derivation.

II. Let $\delta : A \to A$ be a local derivation. Then for any $a \in A$ there exists a derivation D_a such that $\delta(a) : D_a (a)$. By the first part of the theorem D_a is a L-derivation. Hence, δ is a local L-derivation.

III. Since $D(x) = D_a(x) = [a,x]$, then the statement is obvious.

Hence we have the following corollary.

Corollary 1. *If R is a real W*-algebra, then a local derivation $\delta : R \to R$ on R_k is a local L-derivation.*

In [7] Sh. Ayupov and K. Kudaybergenov proved that in a semisimple finite-dimensional Lie algebra, every local L-derivation is an L-derivation. In the infinite-dimensional case, a similar result is available in the work of Y. Yao and K. Zhao [8]. Here, the semi-simplicity of the algebra means that R does not have non-zero abelian ideals. Since the Lie algebra R_k is semisimple, then from the above and by Corollary 1, we get the following result.

Theorem 5. *The restriction δ_{R_k} of δ to the Lie algebra R_k is a L-derivation, i.e., local derivation on elements of the skew-hermitian part of a real W*-algebra satisfies the Leibniz identity for multiplication* [.,.].

The next question naturally emerges: is the L-derivation δ_{R_k} an ordinary derivation? Since in a semisimple Lie algebra every L-derivation is inner, then, by Theorem 4 (item 3), we can state that the restriction of a local derivation of a real W*-algebra to the skew-hermitian part R_k is a derivation, i.e., it satisfies the Leibniz identity. Now the next question arises: If the local derivation δ satisfies the Leibniz identity in both hermitian elements and skew-hermitian elements, does their product satisfy this identity? This question has a positive solution in the following special case.

Theorem 6. *If for local derivation* δ *is true* $\delta|_{R_s} = [a,.]$ *and* $\delta|_{R_k} = [a,.]$ *then* $\delta = [a,.]$ *i.e.,*
$\delta(xy) = \delta(x)y + e\delta(y)$, *for any* $x, y \in R$.

Proof. It is enough to show for elements $x \in R_s, y \in R_k$. Since $xy = x_1 + y_1$, $x_1 \in R_s$, $y_1 \in R_k$, then we get

$$\begin{aligned}\delta(xy) &= \delta(x_1 + y_1) = \delta(x_1) + \delta(y_1) = [a, x_1] + [a, y_1] = ax_1 - x_1a + ay_1 - y_1a = \\ &= a(x_1 + y_1) - (x_1 + y_1)a = [a, x_1 + y_1] = \\ &= [a, xy] = a(xy) - (xy)a = a(xy) - xay + xay - (xy)a = \\ &= [a, x]y + x[a, y] = \delta(x)y + x\delta(y)\end{aligned}$$

ie.

$$\delta(xy) = \delta(x)y + x\delta(y).$$

Next, we will consider local derivations following the works of M. Bresar. First, let us recall the definition of module and bimodule. An A-module is the Abelian group M for which the operation of multiplication $A \times M \to M, (r,m) \to rm$, by elements of the ring A, which satisfies the following conditions

I. $(r_1r_2)m = r_1(r_2m)$

II. $\mathbf{1}m = m$

III. $r(m_1 + m_2) = m_1 + m_2$

IV. $(r_1 + r_2)m = r_1m + r_2m$

for all , $m, m_1, m_2 \in M$, $r, r_1, r_2 \in A$.

In the case of a non-commutative ring, such modules are often called left modules. Right modules are those objects for which, under all conditions, multiplication by elements of the ring is performed on the right. In the case of a commutative ring, the definitions of a right and left module are the same, and they are simply called modules.

If A and B are rings, then an (A,B) -bimodule is an Abelian group M such that 1) M is a right B -module and a left A -module. 2) $(am)b = a\,(mb)$, $\forall a \in A$, $\forall b \in B$, $\forall m \in M$, (A,A)-bimodule is briefly called A-bimodule.

Further, let A -ring, M be an arbitrary A-bimodule, $d : A \to M$ be an additive map. Consider the condition:

$$yz = xy = 0 \;\Rightarrow\; xd(y)z = 0 \quad \forall x, y, z \in A. \qquad (d1)$$

It is not difficult to notice that any derivation satisfies the condition $(d1)$.

Local mappings and property $(d1)$ are closely related. Namely, Let $d : A \to M$ be a local derivation. Then $xd(y)z = xd_y(y)z$, for $\forall x,y,z \in A$. On the other hand, $d_y(xy)z = d_y(x)yz + xd_y(y)z$, therefore, $d_y(xy)z - d_y(x)yz = xd_y(y)z = xd(y)z$, hence $xd\,(y)z = 0$ for $yz = xy = 0$ Therefore, local derivation also satisfies the condition $(d1)$.

Let P be the set of idempotents in A, let R be the subring of A that is generated by the set P, and let I be the ideal A that is generated by the commutator

[P,A]. Then $I \subseteq R$ (see [6]). Let us formulate one auxiliary result from [6].

Theorem 7 [6]. *Let A be a ring with identity, M be an A -bimodule with identity, d: A → M be an additive with* $d(1) = 0$ *and satisfying the condition* $(d1)$. *Then the restriction of map* d *to R is a derivation. In addition, the next equalities hold*

$$d(rx)s + rd(xs) = d(rxs) + rd(x)s, \quad r,s \in R,\ x \in A \quad (**)$$

and

$$I\big(d(x)y + xd(y) - d(xy)\big)I = 0, \quad x,y \in A.$$

Let's consider one more condition:

$$IxI = 0 \ \Rightarrow\ x = 0, \text{ where } x \in A \qquad (d2)$$

Theorem 8. *Suppose that A is a ring with identity, d: A → A is an additive map with* $d(1) = 0$ *and satisfying property* $(d1)$. *If A contains a nontrivial idempotent and satisfies the condition* $(d2)$, *then the mapping,* d *is a derivation.*

Proof. By Theorem 7 *d* is a derivation in *I*. Moreover, for $\forall u,v \in I$, $\forall x \in A$:

$$d(uxv) = d(ux)v + uxd(v) \qquad (1.1)$$

$$d(uxv) = d(u)xv + ud(xv) \qquad (1.2)$$

From (**) it follows

$$d(uxv) + ud(x)v = d(ux)v + ud(xv).$$

Using this and (1.2) we find

$$d(u)xv + ud(xv) + ud(x)v = d(ux)v + ud(xv).$$

Hence $d(ux)v = d(u)xv + ud(x)v$. Substituting this into (1.1) we get,

$$d(uxv) = d(u)xv + ud(x)v + uxd(v) \qquad (1.3)$$

Equating the right-hand sides of (1.1) and (1.4), we obtain $d(ux)v = d(u)xv + ud(x)v$, hence

$$\big([d(u)x + ud(x)] - d(ux)\big)v = 0.$$

Similarly, equating (1.2) and (1.4), we have $u\big(d(xv) - [d(x)v + xd(v)]\big) = 0$. Since from *Ib* = 0 and *bI* = 0 it follows *IbI* = 0, then from the condition $(d2)$ we get

$$d(ux) = d(u)x + ud(x), d(xv) = d(x)v + xd(v), \ \forall x \in A, \forall u,v \in I.$$

Thus $d(xyv) = xyd(v) + d(xy)v$ and

$$d(xyv) = xd(yv) + d(x)yv = xd(y)v + xyd(v) + d(x)yv, \quad x,y \in A,\ v \in I.$$

Equating the last equalities we get: $\big(d(x)y + xd(y) - d(xy)\big)v = 0$, hence $d(xy) = xd(y) + d(x)y, \ \forall x,y \in A$. From Theorem 8 we get

Theorem 9. *Let A be a ring with property* $(d2)$ *and A contain a nontrivial idempotent. Then any local derivation of A is a derivation.*
From Theorem 9 we obtain

Corollary 2. *If A is a real (or complex) von Neumann algebra with the property* $(d2)$*, then every local derivation of A is a derivation. In particular, if A is a prime, then any local derivation is a derivation.*

Corollary 3. *If A is a real or complex factor, then every local derivation of A is a derivation, therefore it is an inner.*

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Conflict of Interest Statement

The authors declare that they have no conflict of interest.

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