

LAPLACE DECOMPOSITION METHOD FOR SOLVING FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS WITH INITIAL VALUE PROBLEMS

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ABSTRACT

Fredholm integro-differential equations (IDEs) are important mathematical models used in various scientific disciplines. However, the presence of nonlinearity in these equations poses significant challenges for conventional numerical methods, such as quadrature formulas. This article presents a comparative analysis of the application of the Laplace decomposition method (LDM), the Adomian decomposition method (ADM), and the homotopy perturbation method (HPM) to nonlinear Fredholm IDEs with initial value problems. All methods were transformed into a sequence of solvable nonlinear integral equations. LDM, a semi-analytical technique specifically designed for nonlinear IDEs, offers an efficient approximation approach. The study presents the results of solving three illustrative examples using LDM and conducts a comparative analysis with other methods, such as ADM and HPM. The results demonstrated that LDM achieved exceptional accuracy and efficiency for nonlinear Fredholm IDEs.

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Introduction

Integro-differential equations (IDEs), both in ordinary and partial forms, play a significant role in mathematical modelling across a wide range of engineering and scientific disciplines. These equations are employed in various fields, including finance, reactor dynamics, population studies, ecology modelling, aerospace engineering, fluid mechanics, biological and chemical systems, process engineering, hydroelectric machinery, and other domains. In this article, linear and nonlinear Fredholm IDEs of the second kind with initial value problems are presented. Conventional numerical methods, such as quadrature formulas, often encounter limitations in providing accurate solutions for these equations due to the presence of nonlinearity. Therefore, there is a pressing need to develop efficient and reliable approximate methods to tackle these challenging equations. To approximate their solutions, researchers have proposed several methods, such as the Adomian decomposition method (ADM) [1-4], Laplace decomposition method (LDM) [5-8], homotopy perturbation method (HPM) [9-10], homotopy analysis method (HAM) [11-13], direct computation method [14], variation iteration method [15], and reproducing kernel method [16]. LDM, which was introduced by Khuri [7], has emerged as a promising computational technique for addressing the nonlinearity inherent in Fredholm IDEs. However, there is a lack of comprehensive research on the application of LDM,

specifically for nonlinear cases with initial value problems. Thus, it is imperative to undertake an exploration and evaluation of the effectiveness of LDM in solving these complex equations.

This article focuses on LDM, which combines the Laplace transform and ADM to approximate the solutions of nonlinear Fredholm IDEs with initial value problems. To evaluate the performance of LDM, a comparative analysis of the results is conducted, focusing on the error comparison between LDM, ADM, and HPM.

Methodology (Implementation of LDM, ADM, and HPM)

In this article, the nonlinear Fredholm IDEs of the second kind are considered in the form:

$$u^{(n)}(x) = f(x) + \int_a^b K(x, t)F(u(t), u'(t))dt \quad (1)$$

with the initial condition:

$$u^{(k)}(0) = \alpha_k, k = 0, 1, 2, \dots, n - 1 \quad (2)$$

where $u^{(n)}(x)$ is the n -th derivative of the unknown function $u(x)$ that will be determined, $K(x, t)$ represents the kernel of the integral equation, $f(x)$ is an analytic function, $F(u(t), u'(t))$ is a nonlinear function, and a and b denote the limits of integration.

By applying the Laplace transform, denoted by \mathcal{L} , to both sides of Equation (2), the following result is obtained:

$$\mathcal{L}[u^{(n)}(x)] = \mathcal{L}[f(x)] + \mathcal{L}\left[\int_a^b K(x, t)F(u(t), u'(t))dt\right] \quad (3)$$

Using the differential property of the Laplace transform, the expression becomes:

$$s^n \mathcal{L}[u(x)] - \sum_{k=1}^n s^{k-1} u^{(n-k)}(0) = \mathcal{L}[f(x)] + \mathcal{L}\left[\int_a^b K(x, t)F(u(t), u'(t))dt\right] \quad (4)$$

Dividing Equation (4) by s^n yields:

$$\begin{aligned} \mathcal{L}[u(x)] - \frac{1}{s^n} \left[\sum_{k=1}^n s^{k-1} \alpha^{(n-k)}(0) \right] \\ = \frac{1}{s^n} \mathcal{L}[f(x)] + \frac{1}{s^n} \mathcal{L}\left[\int_a^b K(x, t)F(u(t), u'(t))dt\right] \end{aligned} \quad (5)$$

Applying the inverse Laplace transform on both sides of Equation (5), the following is derived:

$$u(x) = G(x) + \mathcal{L}^{-1} \left[\frac{1}{s^n} \mathcal{L}\left[\int_a^b K(x, t)F(u(t), u'(t))dt\right] \right] \quad (6)$$

where:

$$G(x) = \mathcal{L}^{-1} \left[\frac{1}{s^n} \left[\sum_{k=1}^n s^{k-1} \alpha^{(n-k)}(0) \right] \right] + \mathcal{L}^{-1} \left[\frac{1}{s^n} [\mathcal{L}[f(x)]] \right] \quad (7)$$

represents the term resulting from the source term and the prescribed initial condition.

The linear term $u(x)$ on the left side is first expressed as an infinite series of components:

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \quad (8)$$

where the components $u_n(x)$, $n \geq 0$ are determined recursively. However, the nonlinear term $F(u(t), u'(t))$ on the right side of Equation (1) will be represented as an infinite series of Adomian polynomials A_n in the form:

$$F(u(t), u'(t)) = \sum_{n=0}^{\infty} A_n(t), \tag{9}$$

where $A_n, n \geq 0$ is defined as:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[F \left(\sum_{i=0}^{\infty} \lambda^i u_i(t), \frac{\partial}{\partial t} \sum_{i=0}^{\infty} \lambda^i u_i(t) \right) \right]_{\lambda=0}, n = 0, 1, 2, 3, \dots \tag{10}$$

Here, A_n can be evaluated for all forms of nonlinearity.

Substituting Equations (8) and (9) into Equation (7) leads to:

$$\sum_{n=0}^{\infty} u_n(x) = G(x) + \mathcal{L}^{-1} \left[\frac{1}{s^n} \mathcal{L} \left[\int_a^b K(x, t) (\sum_{n=0}^{\infty} A_n(t)) dt \right] \right] \tag{11}$$

ADM provides the recursive relation as follows:

$$\begin{aligned} u_0(x) &= G(x) \\ u_{n+1}(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^n} \mathcal{L} \left[\int_a^b K(x, t) A_n(t) dt \right] \right], n \geq 0 \end{aligned} \tag{12}$$

Equation (12) represents the general scheme of LDM for nonlinear Fredholm IDEs given in Equations (1) and (2):

To apply ADM for the IDEs in Equations (1) and (2), they are transformed into integral equations of the Fredholm type:

$$u(x) = g(x) + L^{-1} \left[\int_a^b K(x, t) F(u(t), u'(t)) dt \right] \tag{13}$$

where:

$$g(x) = L^{-1}(f(x)) + u(a) + u'(a)(x - a) + \frac{u''(a)}{2!}(x - a)^2 + \dots + \frac{u^{(N-1)}(a)}{(N-1)!}(x - a)^{N-1} \tag{14}$$

with:

$$L^{-1}(u(x)) = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} u(t) dt \tag{15}$$

Using Equations (8), (9), and (10), the following ADM scheme is obtained:

$$\begin{aligned} u_0(x) &= g(x) \\ u_{n+1}(x) &= L^{-1} \left[\int_a^b K(x, t) A_n(t) dt \right], n \geq 0 \end{aligned} \tag{16}$$

To visually illustrate the fundamental concept of HPM, consider the nonlinear functional equation (Biazar & Ghazvini, 2008) in the form:

$$L(u) + N(u) = f \tag{17}$$

In a convex homotopy form, the perturbation scheme is constructed as follows:

$$H(v, p) = (1 - p)(L(v) - L(u_0)) + p(L(v) + N(v) - f) = 0 \tag{18}$$

where $p \in [0, 1]$ is the homotopy parameter and u_0 is an initial guess satisfying the initial or

boundary conditions in Equations (1) and (2). It is evident that when $p = 0$ and $p = 1$, the following conditions hold:

$$\begin{aligned} H(v, 0) &= (L(v) - L(u_0)) = 0 \\ H(v, 1) &= L(v) + N(v) - f = 0 \end{aligned} \tag{19}$$

in which v varies from the initial u_0 to the exact solution u of Equation (17). Equating the homotopy function in Equation (18) to zero yields the following expression:

$$L(v(x, p)) = L(u_0) + p (f - N(v(x, p)) - L(u_0)) \tag{20}$$

Let the solution to Equation (17) be sought in the form:

$$v(x, p) = \sum_{k=0}^{\infty} v_k(x) p^k \tag{21}$$

Substituting Equation (21) into Equation (20) yields:

$$L(\sum_{k=0}^{\infty} v_k(x) p^k) = L(u_0(x)) + p (f(x) - N(\sum_{k=0}^{\infty} v_k(x) p^k) - L(u_0(x))) \tag{22}$$

By comparing coefficients of terms in Equation (22) with identical powers of P , it results in:

$$\begin{aligned} p^0: v_0(x) &= u_0(x) \\ p^1: v_1(x) &= L^{-1}(f(x)) - L^{-1}(N(v_0(x))) - u_0(x), n \geq 0 \\ p^k: v_k(x) &= -L^{-1}(N(v_{k-1}(x))), \quad k = 2, 3, \dots \end{aligned} \tag{23}$$

where L^{-1} is the inverse operator of L .

Hence, the analytical-approximate solution is expressed as:

$$u = \lim_{p \rightarrow \infty} v(x, p) = v(x) = \sum_{k=0}^{\infty} v_k(x) \tag{24}$$

Applications and Results

This section demonstrates the application of three methods, LDM, ADM, and HPM, using the following examples.

Example 1 [6]: Consider the linear Fredholm IDE given by:

$$u''(x) = e^x - x + \int_0^1 xtu(t)dt \tag{25}$$

with the initial conditions:

$$u(0) = 1, \quad u'(0) = 1 \tag{26}$$

and the exact solution is $u(x) = e^x$.

Method 1 (LDM): To solve Equations (25) and (26) using LDM, the Laplace transform is applied to both sides of Equation (25):

$$s^2 \mathcal{L}[u(x)] - su(0) - u'(0) = \frac{1}{s-1} - \frac{1}{s^2} + \mathcal{L} \left[\int_0^1 xtu(t) dt \right] \tag{27}$$

Dividing both sides of Equation (28) by s^2 and taking the initial conditions from Equation (26) into account yields:

$$\begin{aligned} \mathcal{L}[u(x)] &= \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^2(s-1)} - \frac{1}{s^4} + \frac{1}{s} \mathcal{L} \left[\int_0^1 xtu(t) dt \right] \\ &= \frac{1}{s-1} - \frac{1}{s^4} + \frac{1}{s^2} \mathcal{L} \left[\int_0^1 xtu(t) dt \right] \end{aligned} \tag{28}$$

Applying the inverse Laplace transform on both sides of Equation (28), the following expression is obtained:

$$u(x) = e^x - \frac{x^3}{6} + \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 xtu(t) dt \right] \right] \quad (29)$$

Applying ADM to Equation (29) leads to the following recursive relations:

$$\begin{aligned} u_0(x) &= e^x - \frac{x^3}{6}, \\ u_1(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 xtu_0(t) dt \right] \right] = \frac{29}{6*30} x^3, \\ u_2(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 xtu_1(t) dt \right] \right] = \frac{29}{6*(30)^2} x^3, \\ &\vdots \\ u_n(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 xtu_{n-1}(t) dt \right] \right] = \frac{29}{6*(30)^{n-1}} x^3. \end{aligned} \quad (30)$$

Therefore, the approximate solution of Example 1 can be readily obtained as follows:

$$u(x) = e^x - \frac{x^3}{6} + \frac{29}{6*30} x^3 + \frac{29}{6*(30)^2} x^3 + \dots + \frac{29}{6*(30)^{n-1}} x^3 + \dots \quad (31)$$

Method 2 (ADM): To solve Equations (25) and (26) using ADM, both sides of Equation (25) are integrated twice with respect to x , and the initial conditions from Equation (26) are taken into account, yielding:

$$u(x) = e^x - \frac{x^3}{6} + \int_0^x (x-t) \left[\int_0^1 t\tau u(\tau) d\tau \right] dt \quad (32)$$

Applying ADM to (32), the recursive relation is presented as follows:

$$\begin{aligned} u_0(x) &= e^x - \frac{x^3}{6}, \\ u_1(x) &= \int_0^x (x-t) \left[\int_0^1 t\tau u_0(\tau) d\tau \right] dt = \frac{29}{6*30} x^3, \\ u_2(x) &= \int_0^x (x-t) \left[\int_0^1 t\tau u_1(\tau) d\tau \right] dx = \frac{29}{6*(30)^2} x^3, \\ &\vdots \\ u_n(x) &= \int_0^x (x-t) \left[\int_0^1 t\tau u_{n-1}(\tau) d\tau \right] dx = \frac{29}{6*(30)^{n-1}} x^3, \end{aligned} \quad (33)$$

Therefore, the approximate solution of Example 1 can be readily obtained as follows:

$$u(x) = e^x - \frac{x^3}{6} + \frac{29}{6*30} x^3 + \frac{29}{6*(30)^2} x^3 + \dots + \frac{29}{6*(30)^{n-1}} x^3 + \dots \quad (34)$$

Which is the same as LDM (31):

Method 3 (HPM): To solve Equations (25) and (26) using HPM, they are converted into an integral equation as stated in Equation (32) i.e.:

$$u(x) = e^x - \frac{x^3}{6} + \int_0^x (x-t) \left[\int_0^1 t\tau u(\tau) d\tau \right] dt \quad (35)$$

A homotopy function can be readily constructed as follows:

$$H(u, p) = u(x) - u_0(x) + p \left(u_0(x) - e^x + \frac{x^3}{6} - \int_0^x (x-t) \left[\int_0^1 t \tau u(\tau) d\tau \right] dt \right) = 0 \tag{36}$$

and express the solution in series form:

$$u(x) = \sum_{n=0}^{\infty} v_n p^n \tag{37}$$

Substituting Equation (26) into Equation (25), the recursive relation is presented as follows:

$$\begin{aligned} p^0: v_0(x) &= u_0(x) = e^x - \frac{x^3}{6}, \\ p^1: v_1(x) &= -u_0(x) + e^x - \frac{x^3}{6} + \int_0^x (x-t) \left[\int_0^1 t \tau v_0(\tau) d\tau \right] dt = \frac{29}{6 \cdot 30} x^3, \\ p^2: v_2(x) &= \int_0^x (x-t) \left[\int_0^1 t \tau v_1(\tau) d\tau \right] dt = \frac{29}{6 \cdot (30)^2} x^3, \\ &\vdots \\ p^n: v_n(x) &= \int_0^x (x-t) \left[\int_0^1 t \tau v_{n-1}(\tau) d\tau \right] dt = \frac{29}{6 \cdot (30)^{n-1}} x^3. \end{aligned} \tag{38}$$

Therefore, the approximate solution of Example 1 can be readily obtained as follows:

$$u(x) = e^x - \frac{x^3}{6} + \frac{29}{6 \cdot 30} x^3 + \frac{29}{6 \cdot (30)^2} x^3 + \dots + \frac{29}{6 \cdot (30)^{n-1}} x^3 + \dots \tag{39}$$

From Equations (31), (34), and (39), it can be concluded that LDM, ADM, and HPM provide the same solution accuracy for the linear IDEs in Equations (25) and (26). Table 1 show the numerical results.

Table 1: Comparison of absolute errors for ADM, LDM, and HPM for Example 1 at $n = 5$

x	Exact solution	ErADM ($n = 5$)	ErLDM ($n = 5$)	ErHPM ($n = 5$)
0.1	1.1051709	2.05761×10^{-10}	2.05761×10^{-10}	2.05761×10^{-10}
0.2	1.2214028	1.64609×10^{-9}	1.64609×10^{-9}	1.64609×10^{-9}
0.3	1.3498588	5.55556×10^{-9}	5.55556×10^{-9}	5.55556×10^{-9}
0.4	1.4918247	1.31687×10^{-8}	1.31687×10^{-8}	1.31687×10^{-8}
0.5	1.6487213	2.57202×10^{-8}	2.57202×10^{-8}	2.57202×10^{-8}
0.6	1.8221188	4.44444×10^{-8}	4.44444×10^{-8}	4.44444×10^{-8}
0.7	2.0137527	7.05761×10^{-8}	7.05761×10^{-8}	7.05761×10^{-8}
0.8	2.2255409	1.05350×10^{-7}	1.05350×10^{-7}	1.05350×10^{-7}
0.9	2.4596031	1.50000×10^{-7}	1.50000×10^{-7}	1.50000×10^{-7}
1.0	2.7182818	2.05761×10^{-7}	2.05761×10^{-7}	2.05761×10^{-7}

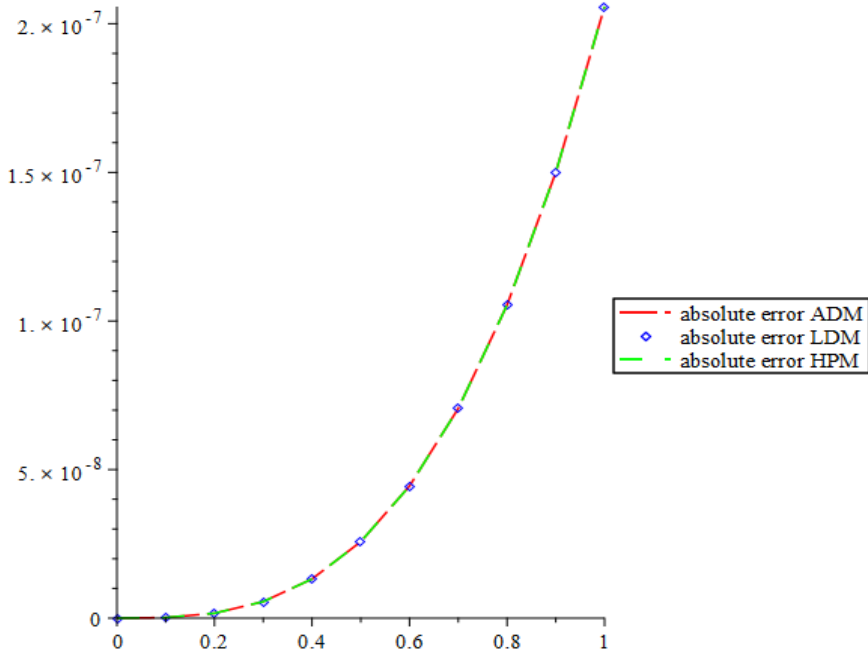


Figure 1: Comparison of absolute errors for ADM, LDM, and HPM for Example 1 at $n = 5$

Table 2: Comparison of absolute errors for ADM, LDM, and HPM for Example 1 at $n = 50$

x	Exact solution	ErADM ($n = 50$)	ErLDM ($n = 50$)	ErHPM ($n = 50$)
0.1	1.1051709	6.96478×10^{-77}	6.96478×10^{-77}	6.96478×10^{-77}
0.2	1.2214028	5.57182×10^{-76}	5.57182×10^{-76}	5.57182×10^{-76}
0.3	1.3498588	1.88049×10^{-75}	1.88049×10^{-75}	1.88049×10^{-75}
0.4	1.4918247	4.45746×10^{-75}	4.45746×10^{-75}	4.45746×10^{-75}
0.5	1.6487213	8.70597×10^{-75}	8.70597×10^{-75}	8.70597×10^{-75}
0.6	1.8221188	1.50439×10^{-74}	1.50439×10^{-74}	1.50439×10^{-74}
0.7	2.0137527	2.38892×10^{-74}	2.38892×10^{-74}	2.38892×10^{-74}
0.8	2.2255409	3.56597×10^{-74}	3.56597×10^{-74}	3.56597×10^{-74}
0.9	2.4596031	5.07732×10^{-74}	5.07732×10^{-74}	5.07732×10^{-74}
1.0	2.7182818	6.96478×10^{-74}	6.96478×10^{-74}	6.96478×10^{-74}

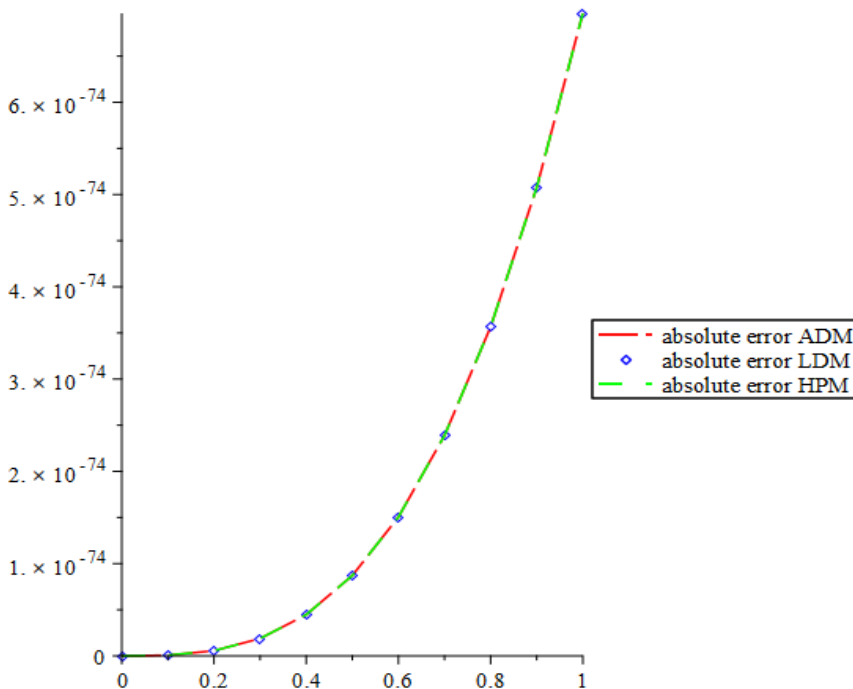


Figure 2: Comparison of absolute errors for ADM, LDM, and HPM for Example 1 at $n = 50$

Remark 1. The results presented in Table 1 to 2, as well as Figure 1 to 2, provide a comprehensive comparison of ADM, LDM, and HPM in solving the second-order linear Fredholm IDEs in Equations (25) and (26). Notably, the analysis reveals that all three methods produce identical absolute error values across different iteration counts (five and 50 iterations), indicating that they achieve the same level of accuracy in approximating the solution. These results highlight the convergence behaviour of the methods, with the absolute error decreasing as the number of iterations increases, approaching the exact solution.

Al-Hayani [6] considered the same IDEs and solved them using LDM, comparing the results with those obtained from HPM and the variational iteration method, concluding that LDM outperformed the others. However, careful investigation shows that both HPM and LDM yield identical errors.

Example 2 [10]: Consider the following nonlinear Fredholm IDE:

$$u'(x) = \cos(x) - \frac{\pi x}{48} + \frac{1}{24} \int_0^\pi x u^2(t) dt, \tag{40}$$

with the initial condition:

$$u(0) = 0, \tag{41}$$

and the exact solution is $u(x) = \sin(x)$.

Method 1 (LDM): To solve Equations (40) and (41) using LDM, the Laplace transform is applied to both sides of Equation (31):

$$s\mathcal{L}[u(x)] - u(0) = \frac{s}{s^2+1} - \frac{\pi}{48s^2} + \mathcal{L}\left[\frac{1}{24}\int_0^\pi xu^2(t) dt\right] \quad (42)$$

Using the initial condition (41) and dividing both sides of Equation (40) by s , the following result is obtained:

$$\mathcal{L}[u(x)] = \frac{1}{s^2+1} - \frac{\pi}{48s^3} + \frac{1}{s}\mathcal{L}\left[\frac{1}{24}\int_0^\pi xu^2(t) dt\right] \quad (43)$$

Applying the inverse Laplace transform on both sides of Equation (42) yields:

$$u(x) = \sin(x) - \frac{\pi x^2}{96} + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{1}{24}\int_0^\pi xu^2(t) dt\right]\right] \quad (44)$$

Applying ADM and considering the nonlinear term representations (9) and (10) result in the following recursive relations:

$$\begin{aligned} u_0(x) &= \sin(x) - \frac{\pi x^2}{2 \cdot 48}, \\ u_1(x) &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{1}{24}\int_0^\pi xA_0(t) dt\right]\right] = \left(\frac{7}{12}\pi - \frac{1}{48}\pi^3 + \frac{1}{46080}\pi^7\right)\frac{x^2}{48}, \\ u_2(x) &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{1}{24}\int_0^\pi xA_1(t) dt\right]\right] = \left(-\frac{7}{72}\pi + \frac{1}{36}\pi^3 - \frac{1}{1152}\pi^5\right. \\ &\quad \left.- \frac{1}{18432}\pi^7 + \frac{1}{368640}\pi^9 - \frac{1}{530841600}\pi^{13}\right)\frac{x^2}{48} \\ &\quad \vdots \\ u_n(x) &= \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{1}{24}\int_0^\pi xA_{n-1}(t) dt\right]\right], \end{aligned} \quad (45)$$

Therefore, the approximate solution of Example 2 can be readily obtained by:

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + \dots + u_n(x) \\ &= \sin(x) - \frac{\pi x^2}{2 \cdot 48} + \left(\frac{7}{12}\pi - \frac{1}{48}\pi^3 + \frac{1}{46080}\pi^7\right)\frac{x^2}{48} \\ &\quad + \left(-\frac{7}{72}\pi + \frac{1}{36}\pi^3 - \frac{1}{1152}\pi^5 - \frac{1}{18432}\pi^7 + \frac{1}{368640}\pi^9 - \frac{1}{530841600}\pi^{13}\right)\frac{x^2}{48} + \dots \end{aligned} \quad (46)$$

Method 2 (ADM): To solve Equations (40) and (41) using ADM, both sides of (40) are integrated with respect to x , taking the initial condition (41) into account, to obtain:

$$u(x) = \sin(x) - \frac{\pi x^2}{96} + \int_0^x \left(\frac{1}{24} \int_0^\pi x u^2(t) dt \right) dx. \tag{47}$$

Applying ADM results in the following recursive relation:

$$\begin{aligned} u_0(x) &= \sin(x) - \frac{\pi x^2}{2 \cdot 48}, \\ u_1(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_0(t) dt = \left(\frac{7}{12} \pi - \frac{1}{48} \pi^3 + \frac{1}{46080} \pi^7 \right) \frac{x^2}{48}, \\ u_2(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_1(t) dt = \left(-\frac{7}{72} \pi + \frac{1}{36} \pi^3 - \frac{1}{1152} \pi^5 \right. \\ &\quad \left. - \frac{1}{18432} \pi^7 + \frac{1}{368640} \pi^9 - \frac{1}{530841600} \pi^{13} \right) \frac{x^2}{48} \\ &\quad \vdots \\ u_n(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_{n-1}(t) dt, \end{aligned} \tag{48}$$

Therefore, the approximate solution of Example 2 can be expressed as:

$$\begin{aligned} u(x) &= u_0(x) + u_1(x) + u_2(x) + \dots + u_n(x) \\ &+ \sin(x) - \frac{\pi x^2}{2 \cdot 48} + \left(\frac{7}{12} \pi - \frac{1}{48} \pi^3 + \frac{1}{46080} \pi^7 \right) \frac{x^2}{48} \\ &+ \left(-\frac{7}{72} \pi + \frac{1}{36} \pi^3 - \frac{1}{1152} \pi^5 - \frac{1}{18432} \pi^7 + \frac{1}{368640} \pi^9 - \frac{1}{530841600} \pi^{13} \right) \frac{x^2}{48} + \dots \end{aligned} \tag{49}$$

Method 3 (HPM): To solve Equation (40) and (41) using HPM, both sides of Equation (40) are integrated with respect to x , taking into account the initial condition (41):

$$u(x) = \sin(x) - \frac{\pi x^2}{96} + \int_0^x \left(\frac{1}{24} \int_0^\pi t u^2(\tau) d\tau \right) dt. \tag{50}$$

A homotopy function can be constructed as follows:

$$\begin{aligned} H(u, p) &= u(x) - u_0(x) \\ &+ p \left(u_0(x) - \sin(x) + \frac{\pi x^2}{96} - \int_0^x \left(\frac{1}{24} \int_0^\pi t u^2(\tau) d\tau \right) dt \right) = 0 \end{aligned} \tag{51}$$

The solution to Equation (40) can be sought in series form as follows:

$$u(x) = \sum_{n=0}^\infty v_n p^n \tag{52}$$

Substituting Equation (52) into Equation (51) yields the following recursive relation:

$$\begin{aligned} p^0: v_0(x) &= u_0(x) = \sin(x) - \frac{\pi x^2}{96}, \\ p^1: v_1(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_0(t) dt = \left(\frac{7}{12} \pi - \frac{1}{48} \pi^3 + \frac{1}{46080} \pi^7 \right) \frac{x^2}{48} \end{aligned}$$

$$\begin{aligned}
 p^2: v_2(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_1(t) dt = \left(-\frac{7}{72} \pi + \frac{1}{36} \pi^3 - \frac{1}{1152} \pi^5 \right. \\
 &\quad \left. - \frac{1}{18432} \pi^7 + \frac{1}{368640} \pi^9 - \frac{1}{530841600} \pi^{13} \right) \frac{x^2}{48} \\
 &\quad \vdots \\
 p^n: v_n(x) &= \int_0^x \frac{1}{24} \int_0^\pi x A_{n-1}(t) dt \tag{53}
 \end{aligned}$$

Therefore, the approximate solution of Example 2 can be represented as follows:

$$\begin{aligned}
 u(x) &= u_0(x) + u_1(x) + u_2(x) + \dots + u_n(x) \\
 &+ \sin(x) - \frac{\pi x^2}{2 \cdot 48} + \left(\frac{7}{12} \pi - \frac{1}{48} \pi^3 + \frac{1}{46080} \pi^7 \right) \frac{x^2}{48} \\
 &+ \left(-\frac{7}{72} \pi + \frac{1}{36} \pi^3 - \frac{1}{1152} \pi^5 - \frac{1}{18432} \pi^7 + \frac{1}{368640} \pi^9 - \frac{1}{530841600} \pi^{13} \right) \frac{x^2}{48} + \dots \tag{54}
 \end{aligned}$$

Table 3: Comparison of absolute errors for ADM, LDM, and HPM for Example 2 at $n = 5$

x	Exact solution	ErADM ($n = 5$)	ErLDM ($n = 5$)	ErHPM ($n = 5$)
0.1	0.0998334	3.61575×10^{-6}	3.61575×10^{-6}	3.61575×10^{-6}
0.2	0.1986693	1.44630×10^{-5}	1.44630×10^{-5}	1.44630×10^{-5}
0.3	0.2955202	3.25418×10^{-5}	3.25418×10^{-5}	3.25418×10^{-5}
0.4	0.3894183	5.78521×10^{-5}	5.78521×10^{-5}	5.78521×10^{-5}
0.5	0.4794255	9.03938×10^{-5}	9.03938×10^{-5}	9.03938×10^{-5}
0.6	0.5646425	1.30167×10^{-4}	1.30167×10^{-4}	1.30167×10^{-4}
0.7	0.6442177	1.77172×10^{-4}	1.77172×10^{-4}	1.77172×10^{-4}
0.8	0.7173561	2.31408×10^{-4}	2.31408×10^{-4}	2.31408×10^{-4}
0.9	0.7833269	2.92876×10^{-4}	2.92876×10^{-4}	2.92876×10^{-4}
1.0	0.8414710	3.61575×10^{-4}	3.61575×10^{-4}	3.61575×10^{-4}

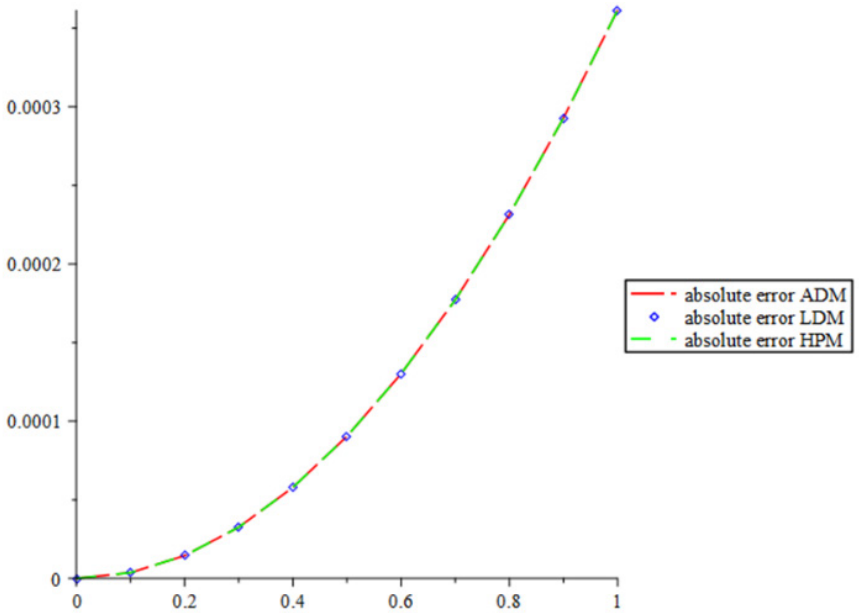


Figure 3: Comparison of absolute errors for ADM, LDM, and HPM for Example 2 at $n = 5$

Table 4: Comparison of absolute errors for ADM, LDM, and HPM for Example 2 at $n = 50$

x	Exact solution	ErADM ($n = 50$)	ErLDM ($n = 50$)	ErHPM ($n = 50$)
0.1	0.0998334	4.30827×10^{-20}	4.30827×10^{-20}	4.30827×10^{-20}
0.2	0.1986693	1.72331×10^{-19}	1.72331×10^{-19}	1.72331×10^{-19}
0.3	0.2955202	3.87745×10^{-19}	3.87745×10^{-19}	3.87745×10^{-19}
0.4	0.3894183	6.89324×10^{-19}	6.89324×10^{-19}	6.89324×10^{-19}
0.5	0.4794255	1.07707×10^{-18}	1.07707×10^{-18}	1.07707×10^{-18}
0.6	0.5646425	1.55098×10^{-18}	1.55098×10^{-18}	1.55098×10^{-18}
0.7	0.6442177	2.11105×10^{-18}	2.11105×10^{-18}	2.11105×10^{-18}
0.8	0.7173561	2.75729×10^{-18}	2.75729×10^{-18}	2.75729×10^{-18}
0.9	0.7833269	3.48970×10^{-18}	3.48970×10^{-18}	3.48970×10^{-18}
1.0	0.8414710	4.30827×10^{-18}	4.30827×10^{-18}	4.30827×10^{-18}

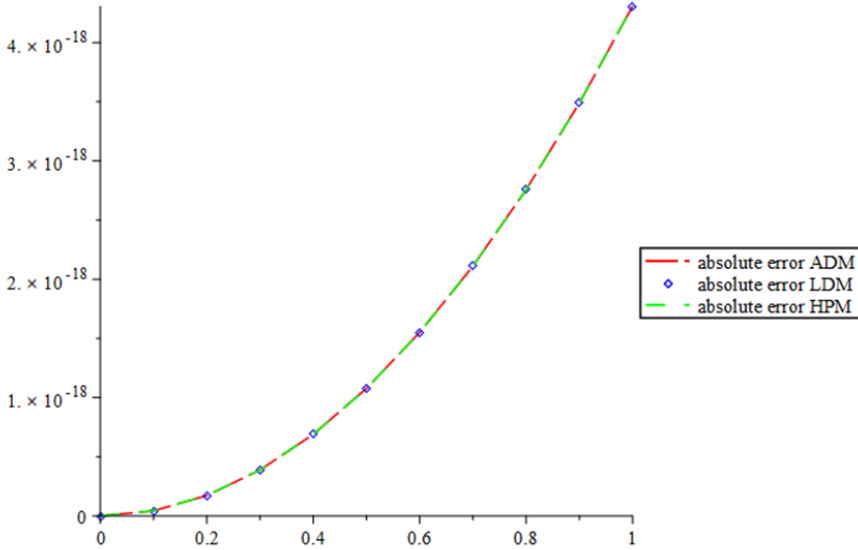


Figure 4: Comparison of absolute errors for ADM, LDM, and HPM for Example 2 at $n = 50$

Remark 2. The results presented in Table 3 to 4, as well as Figure 3 to 4, provide a detailed comparison of ADM, LDM, and HPM in solving a first-order nonlinear Fredholm IDE. The analysis indicates that all three methods produce identical absolute error values across different iterations (five and 50 iterations), demonstrating that they achieve the same level of accuracy in approximating the solution. These findings underscore the convergence behaviour of the methods, wherein the absolute error decreases as the number of iterations increases, approaching the exact solution.

Saha *et al.* [10] also solved Example 2 using HPM, ADM, and the series solution method, reporting that HPM yielded the exact solution, while the other methods exhibited some errors. A detailed investigation reveals that ADM can provide a more accurate solution if the initial guess is selected appropriately.

Example 3 [7]: The following nonlinear Fredholm IDE is considered:

$$u''(x) = \sinh(x) + x - \int_0^1 x(\cosh^2(t) - u^2(t))dt, \tag{55}$$

with the initial condition:

$$u(0) = 0, \quad u'(0) = 1, \tag{56}$$

and the exact solution is $u(x) = \sinh(x)$

Method 1 (LDM): To solve Equations (55) and (56) using LDM, the Laplace transform is applied on both sides of Equation (55):

$$s^2 \mathcal{L}[u(x)] - su(0) - u'(0) = \frac{1}{s^2-1} + \frac{1}{2s^2} - \frac{\cosh(1)\sinh(1)}{2s^2} + \mathcal{L}\left[\int_0^1 x(u^2(t))dt\right] \tag{57}$$

Dividing both sides of Equation (57) by s^2 and incorporating the initial conditions (56) result in:

$$\mathcal{L}[u(x)] = \frac{1}{s^2} + \frac{1}{s^2(s^2-1)} + \frac{1}{2s^4} - \frac{\cosh(1)\sinh(1)}{2s^4} + \frac{1}{s^2} \mathcal{L} \left[\int_0^1 x(u^2(t))dt \right] \tag{58}$$

Applying the inverse Laplace transform to both side of Equation (58) yields:

$$u(x) = \sinh(x) + \frac{x^3}{12} (1 - \cosh(1) \sinh(1)) + \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 x(u^2(t))dt \right] \right] \tag{59}$$

Applying ADM results in the following recursive relation:

$$\begin{aligned} u_0(x) &= \sinh(x) - 0.06778585037x^3, \\ u_1(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 x(A_0(t))dt \right] \right] = 0.06281687946x^3, \\ u_2(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 x(A_1(t))dt \right] \right] = 0.004503342225x^3, \\ &\vdots \\ u_n(x) &= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \left[\int_0^1 x(A_{n-1}(t))dt \right] \right] \end{aligned} \tag{60}$$

Therefore, the approximate solution to Example 3 can be expressed as:

$$\begin{aligned} u(x) &= \sinh(x) - 0.06778585037x^3 + 0.06281687946x^3 \\ &\quad + 0.004503342225x^3 + \dots \end{aligned} \tag{61}$$

Remark 3. Example 3 was solved by Manafianheris [7] using LDM, with the claim that the exact solution can be achieved within two iterations. However, investigations reveal that the exact solution can be obtained by increasing the number of iterations.

Method 2 (ADM): To solve Equations (55) and (56) using ADM, both sides of Equation (55) are integrated twice with respect to x, while taking the initial conditions (56) into account. This yields:

$$\begin{aligned} u(x) &= \sinh(x) + \frac{x^3}{12} (1 - \cosh(1) \sinh(1)) \\ &\quad + \int_0^x (x-t) \left[\int_0^1 t(u^2(\tau))d\tau \right] dt \end{aligned} \tag{62}$$

Applying ADM leads to the following recursive relation:

$$\begin{aligned} u_0(x) &= \sinh(x) - 0.06778585037x^3, \\ u_1(x) &= \int_0^x (x-t) \left[\int_0^1 tA_0(t)d\tau \right] dt = 0.06281687946x^3, \\ u_2(x) &= \int_0^x (x-t) \left[\int_0^1 tA_1(t)d\tau \right] dt = 0.004503342225x^3, \\ &\vdots \\ u_n(x) &= \int_0^x (x-t) \left[\int_0^1 tA_{n-1}(t)d\tau \right] dt \end{aligned} \tag{63}$$

Therefore, the approximate solution to Example 4 can be obtained as follows:

$$u(x) = \sinh(x) - 0.06778585037x^3 + 0.06281687946x^3 + 0.004503342225x^3 + \dots \tag{64}$$

Method 3 (HPM): To solve Equations (55) and (56) using HPM, IDE (55) is reduced to an integral equation of the following form:

$$u(x) = \sinh(x) + \frac{x^3}{12}(1 - \cosh(1) \sinh(1)) + \int_0^x (x-t) \left[\int_0^1 t(u^2(\tau))d\tau \right] dt \tag{65}$$

A homotopy function can then be readily constructed as follows:

$$H(u, p) = u(x) - u_0(x) + p \left(u(x) - \sinh(x) - \frac{x^3}{12}(1 - \cosh(1) \sinh(1)) + \int_0^x (x-t) \left[\int_0^1 t(u^2(\tau))d\tau \right] dt \right) = 0 \tag{66}$$

Seeking for the solution leads to:

$$u(x) = \sum_{n=0}^{\infty} v_n p^n \tag{67}$$

Substituting Equation (56) into Equation (55) and deriving the recursive relation yields:

$$\begin{aligned} p^0: v_0(x) &= u_0(x) = \sinh(x) - 0.06778585037x^3, \\ p^1: v_1(x) &= -u_0(x) + \sinh(x) - 0.06778585037x^3 \\ &\quad + \int_0^x (x-t) \left[\int_0^1 tA_0(t)d\tau \right] dt = 0.06281687946x^3, \\ p^2: v_2(x) &= \int_0^x (x-t) \left[\int_0^1 tA_2(t)d\tau \right] dt = 0.004503342225x^3, \\ &\quad \vdots \\ p^2: v_n(x) &= \int_0^x (x-t) \left[\int_0^1 tA_{n-1}(t)d\tau \right] dt. \end{aligned} \tag{68}$$

Therefore, the approximate solution to Example 4 can be readily obtained by:

$$u(x) = \sinh(x) - 0.06778585037x^3 + 0.06281687946x^3 + 0.004503342225x^3 + \dots \tag{69}$$

The numerical results for the proposed methods (LDM, ADM, and HPM) are presented as follows:

Table 5: Comparison of absolute errors for ADM, LDM, and HPM for Example 4 at $n = 5$

x	Exact solution	ErADM ($n = 5$)	ErLDM ($n = 5$)	ErHPM ($n = 5$)
0.1	0.1001668	5.48149×10^{-9}	5.48149×10^{-9}	5.48149×10^{-9}
0.2	0.2013360	4.38519×10^{-8}	4.38519×10^{-8}	4.38519×10^{-8}
0.3	0.3045203	1.48000×10^{-7}	1.48000×10^{-7}	1.48000×10^{-7}
0.4	0.4107523	3.50815×10^{-7}	3.50815×10^{-7}	3.50815×10^{-7}
0.5	0.5210953	6.85186×10^{-7}	6.85186×10^{-7}	6.85186×10^{-7}
0.6	0.6366536	1.18400×10^{-6}	1.18400×10^{-6}	1.18400×10^{-6}
0.7	0.7585837	1.88015×10^{-6}	1.88015×10^{-6}	1.88015×10^{-6}
0.8	0.8881060	2.80652×10^{-6}	2.80652×10^{-6}	2.80652×10^{-6}
0.9	1.026517	3.99601×10^{-6}	3.99601×10^{-6}	3.99601×10^{-6}
1.0	1.175201	5.48149×10^{-6}	5.48149×10^{-6}	5.48149×10^{-6}

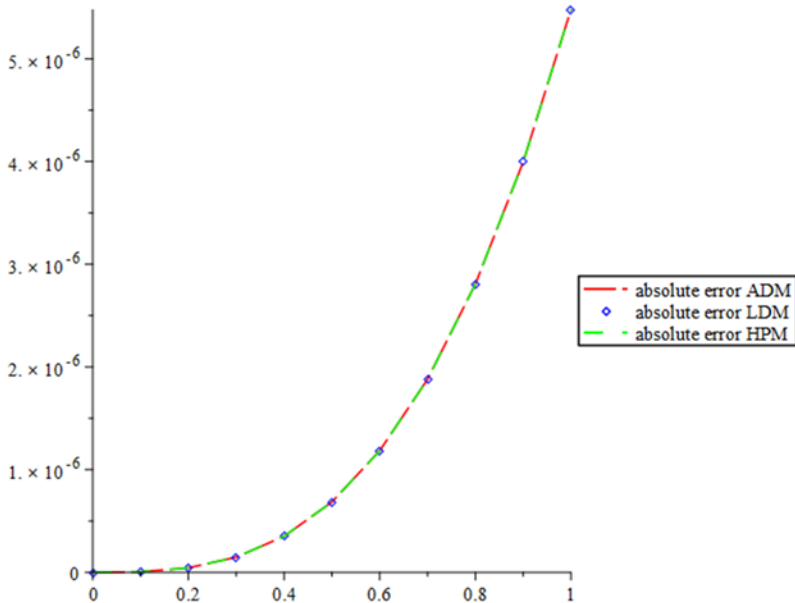


Figure 5: Comparison of absolute errors for ADM, LDM, and HPM for Example 4 at $n = 5$

Table 6: Comparison of absolute errors for ADM, LDM, and HPM for Example 4 at $n = 30$

x	Exact solution	ErADM ($n = 30$)	ErLDM ($n = 30$)	ErHPM ($n = 30$)
0.1	0.1001668	1.53411×10^{-16}	2.79683×10^{-17}	1.37074×10^{-16}
0.2	0.2013360	6.67653×10^{-15}	1.07564×10^{-16}	5.79600×10^{-15}
0.3	0.3045203	1.02045×10^{-14}	2.02999×10^{-15}	9.95450×10^{-15}
0.4	0.4107523	2.26375×10^{-14}	3.86927×10^{-15}	2.04400×10^{-14}
0.5	0.5210953	1.74036×10^{-14}	1.63805×10^{-15}	1.52896×10^{-14}
0.6	0.6366536	3.61642×10^{-14}	1.16646×10^{-14}	3.32264×10^{-14}
0.7	0.7585837	2.82630×10^{-14}	6.81379×10^{-15}	2.60818×10^{-14}
0.8	0.8881060	6.25166×10^{-14}	1.72000×10^{-14}	5.31590×10^{-14}
0.9	1.026517	4.87095×10^{-14}	1.06496×10^{-15}	3.59300×10^{-14}
1.0	1.175201	8.06670×10^{-14}	3.09444×10^{-14}	7.53000×10^{-14}

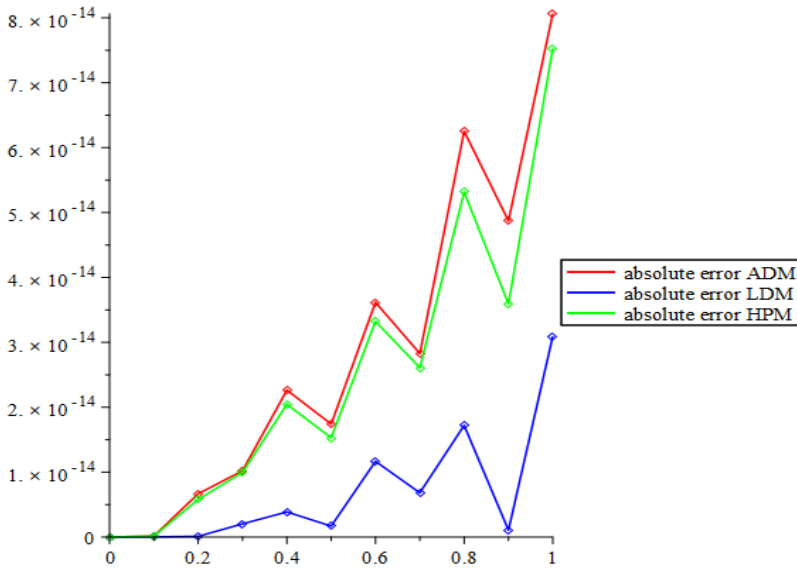


Figure 8: Comparison of absolute errors for ADM, LDM, and HPM for Example 4 at $n = 30$

Remark 4. The results from Example 3 offer insightful comparisons among ADM, LDM, and HPM in solving second-order nonlinear Fredholm IDEs. As shown in Table 5 and Figure 5, these methods initially produce similar absolute error values. However, as the number of iterations increases, LDM outperforms both ADM and HPM, exhibiting a lower absolute error at 30 iterations. In contrast, ADM shows higher absolute error than HPM and LDM at 30 iterations. In conclusion, while ADM, LDM, and HPM demonstrate comparable performance at lower iterations, LDM proves superior in terms of accuracy for higher-order equations.

Conclusions

This study demonstrates the effectiveness of ADM, LDM, and HPM in solving linear and nonlinear Fredholm IDEs with initial value problems. The accuracy of the solutions improves as the number of iterations increases, resulting in a decrease in the absolute error. Notably, LDM outperforms both ADM and HPM, exhibiting superior performance with lower absolute error at higher iterations. This finding underscores the unique computational strategies of LDM, which contribute to its enhanced accuracy and efficiency in solving higher-order equations compared with the other methods. This study contributes to the field of numerical methods for solving linear and nonlinear Fredholm IDEs, highlighting LDM as a robust and effective approach, validated through comparisons with ADM and HPM. Further research may explore the applicability of LDM in diverse mathematical and scientific contexts, as well as its performance in more complex scenarios.

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Conflicts of Interest Statement

The authors declare that they have no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript or in the decision to publish the results. The submitting authors are responsible for ensuring that co-authors declaring their interests.

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