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A STUDY ON SOME PROPERTIES FOR WEAK STABILITY OF NON-AUTONOMOUS DISCRETE DYNAMICAL SYSTEMS

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ABSTRACT

The interest in non-autonomous discrete dynamical systems has grown in recent years due to their relevance in describing real-world activities. For instance, external factors disturb the mapping in each iteration. In mathematics, stability theory explores the stability of solutions and trajectories of dynamical systems under slight perturbations of initial conditions. This paper introduces the concept of weak stability for non-autonomous discrete dynamical systems (NADDS). Analysis is also performed on a set of weak stable points, and the connection between weak stability and shadowing property is investigated. Additionally, this study also considers the connection between weak stability of a NADDS and its induced system.

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INTRODUCTION

In essence, anything that evolves over time can be thought of as a dynamical system, as noted by [33]. Take, for example, the swaying motion of sugar dissolving in a cup of coffee. A dynamical system comprises two components: A state vector that accurately depicts the status of a given real or hypothetical system, and a mapping (or an evolution rule) that informs us, given a current state, about the state the system will assume in the next moment of time [33]. Mathematically, a dynamical system is presented as an initial value problem. A dynamical system is also deterministic if it yields a unique outcome for every state, or stochastic (random) if it involves a probability distribution of potential outcomes to every state, as described by [23].

The concept of a dynamical system originates from Newtonian mechanics. As in other natural sciences and engineering disciplines, the mapping of dynamical systems is an implicit relation — either difference or differential equations or other time scale — that provides the states of the system for only a brief period into the future. The states of all subsequent periods are determined by iteratively applying the relation many time, each increment of time being a small step size. This iterative process is akin to interpreting (or integrating) the dynamical system. If the dynamical system can be interpreted, starting from an initial point, it becomes possible to predict all its future states. A Set of vectors (or points) is termed a trajectory or orbit. This elucidates why a dynamical system is perceived as an initial value problem.

In tracing the historical development of dynamical systems, initial theories focused on the long-term qualitative characteristic of dynamical systems and the stability of solutions of trajectories small disturbances of initial conditions, commonly referred to as the stability of a dynamical system. Subsequently, the emergence of applications for certain types of dynamical systems addressed real-life problems. The study of dynamical systems centres on dynamical systems theory, finding applications in a broad variety of fields, such as medicine, history, economics, engineering, chemistry, biology, physics, and mathematics. Dynamical systems theory, as a principle, deals with the long-term qualitative characteristics of dynamical systems, and explores the properties of solutions of dynamical systems that manifest in real-world scenarios.

Moreover, dynamical systems theory describes the process of evolution and variations, rather than focusing solely on evolutionary outcomes. Essentially, in dynamical system terms, there is no end point of evolution, as outlined by [32]. Additionally, with its emphasis on variation and the speed of change, dynamical systems theory addresses issues related to variation from one time point to the next and overall forms of variation [32]. The accomplishments of dynamical systems theory encompass a collection of ideas that facilitate the examination of overall forms of variations, such as self-regulation, destabilisation and stabilisation [32]. Daghar and Marzougui [8] studied the dynamics of monotone maps on regular curves. Their study delved into the nonwandering set of monotone maps and the structure of ω-limit (resp. α-limit) sets for the maps. It was noted that every infinite minimal set is a Cantor set, which is an adding machine.

A discrete dynamical system is characterised by the restriction of time to the set of integers or a fixed interval, resulting in a sequence of states. In short, a non-autonomous discrete dynamical system (NADDS) is a couple $(T,\mathcal{H})$, where T is a topological space (example.g., a compact metric space), and $\mathcal{H}$ is a sequence of continuous mappings $(h_1,h_2,...)$ from T to itself, at outlined by [1]. Essentially, a NADDS can be viewed as a topological dynamic system given its combination of a topological space (which is T) and a continuous mapping (which is $\mathcal{H}$) within that space. A topological dynamical system is the main focus of research in topological dynamics, a branch of the principles of dynamical systems exploring asymptotic and qualitative characteristics from a topological standpoint. George David Birkhoff is credited as the founder of this field.

Numerous natural systems occurring in nature have been studied using mathematical models. For instance, systems like the logistic model have been used to characterise population growth, while continuous systems (where time is confined to real numbers), such as the Lorenz model, have been instrumental in achieving precise weather predictions. Several mathematical models exploring such systems have been proposed, and the long-term characteristics of such systems have been investigated. However, most mathematical models are autonomous, meaning they involve only a unique continuous mapping that maps a topological space to itself. Consequently, these models cannot be used to represent a general dynamical system.

The interest in NADDS has surged in recent years because they prove suitable for describing real-world activities. For instance, situations where the mapping is disturbed in each iteration due to external factors [19]. NADDS are also valuable for modelling phenomena in physics, biology (e.g. the evolution of populations) and economics, as well as for solving problems in mathematics. This heightened interest stems from the complexity of real-life problems, making it challenging to find a unique function or evolution rule to determine all states in a given problem. Hence, NADDS emerge as a natural choice for studying such real-life problems, offering a sequence of continuous mappings $\mathcal{H}=(h_1,h_2,...)$ capable of describing various situations in a real-life context.

The primary objective of dynamical systems theory is to uncover the characteristics of all orbits. However, due to the inherent complexity of NADDS, delving into pseudo-orbits becomes for gaining a better understanding of true orbits. In the forthcoming sections of this paper, particular attention will be devoted to the investigation of shadowing property.

The concept of weak stability, as explored by Honary and Bahabadi [14], revealed that the orbit's shadowing property is a common feature among weak stable homeomorphisms. Motivated by this insight, this study's focus is directed towards exploring weak stability within the realm of NADDS. A key aspect of this exploration is to scrutinize whether a connection between weak stability and the shadowing property.

Lastly, NADDS introduce an induced system in the space of compact subsets. Investigating the link between dynamical characteristics, such as weak stability, in the original systems and their induced counterparts promises insights into their intricate interplay.

LITERATURE REVIEW

Strong Development of NADDS

As highlighted in the introduction, there is a strong demand for studying and constructing the theory of NADDS, as they serve as suitable mathematical models for describing real-life phenomena and addressing various mathematical problems. Consequently, researchers have been drawn to the rich dynamics of NADDS [19], and they have made significant contributions to understanding the dynamical characteristics of NADDS. Notable results include investigations into entropy [17], chaos (considering various types) [9, 34], Li-Yorke chaos [7], mixing properties [2, 25], sensitivity [15, 24] and shadowing properties [30].

Weak Stability and Shadowing Property

The primary role of dynamical systems theory is to uncover the characteristics of all orbits. However, due to the inherent complexity of NADDS, the study of pseudo-orbits becomes pivotal for gaining a better understanding of true orbits. Pseudo-orbits, such as those generated computationally with round-off errors during the calculation of a point's orbit, serve as typical examples [10]. Given that computers are commonly employed for dynamical systems computations, a thorough comprehension of pseudo-orbits becomes essential. Furthermore, the investigation of the shadowing property is an active subject of research among scholars [28, 29, 14, 31, 26, 11, 12, 16, 30, 38].

In a pivotal study, Honary and Bahabadi [14] explored a discrete dynamical system comprising a compact metric space (M,d) and a homeomorphism $g: M\to M$. This current study delves into the concept of weak stability and found that the orbit's shadowing property is a common feature among weak stable homeomorphisms. Inspired by this insight, this paper discusses the weak stability of a sequence of the continuous mappings from a compact metric space to itself in NADDS and examines the connection between weak stability (of a sequence of the continuous mappings) and the shadowing property.

NADDS and Its Induced System

In the domain of topological dynamical systems, NADDS give rise to two distinct induced systems—one on the hyperspace, representing a deterministic system, and another on the probability measures space, constituting a stochastic system [24]. The hyperspace, defined as a topological space comprising all closed subspaces of another topological space, is equipped with specific topologies like the Hausdorff metric and Vietoris topology. A natural question arises: What connections exist among the dynamic characteristics of the original systems and their induced counterparts? Several researchers have extensively delved into the exploration of interactions between the original systems and their corresponding induced systems [19]. Firstly, properties may be inherited bidirectionally, where the dynamical characteristics of induced mappings inherit those of the original mappings, and vice versa, or both mappings mutually inherit properties from each other [4, 6, 10, 13, 18, 20, 21, 22]. Secondly, induced mappings may not necessarily inherit properties from the original mappings [4, 22]. Thirdly, certain properties of the original mappings can be deduced from the properties of the induced mappings, and vice versa. Additionally, properties of both the original and induced mappings can be deduced from each other [3, 27, 35, 36]. Fourthly, the properties of the induced mappings can also be deduced from the properties of both the original and induced mappings [10]. These insights enable a more comprehensive examination of the connection between the weak stability of NADDS and their induced systems. For instance, it becomes possible to investigate and establish that weakly stable continuous mappings in the original system imply weak stability in the induced system. Furthermore, it has been observed that the hyperspace in the induced system is compact if and only if the metric space in the original system is compact, and if g is continuous, then so is $\overline{g}$ [5]. Lastly, it's worth noting that the assertion by Bernardes Jr. *et al.* [6] stating that $\overline{g}$ is Devaney chaotic if and only if g is Devaney chaotic, is not universally true in the context of nonlinear dynamics. This underscores that the types of spaces in dynamic systems can influence the inheritance of properties between the original and induced mappings.

WEAK STABILITY AND SHADOWING PROPERTY OF NADDS

The definitions of weak stability and shadowing property are adopted from Lan and Peris [19]. This utilisation of certain definitions, lemmas, propositions, and theorems from this source contributes significantly to the objectives of this paper.

Weak Stability of NADDS

Definition 3.1 [19]. A NADDS is defined as a pair $(T,\mathcal{H})$ where $\mathcal{H}=\{h_n\}_{n=1}^{\infty}$ is a sequence of continuous mappings $h_n: T\rightarrow T$, operating on a compact metric space (T,d). For each $j,n\in\mathbb{N}$ with $j\leq n$, $h_j^n = h^n \circ h_{n-1} \circ\ldots\circ h_j$, where h_j^0 is the identity map id_T on T. The associated NADDS is denoted by $\mathcal{H}_p = \{h_{jp+1}^{jp+p}\}_{j=0}^{\infty}$, p ≥ 1.

Definition 3.2 [19]. The orbit of a point $x\in T$ for a NADDS $(T,\mathcal{H})$ is represented by $o(x,\mathcal{H})$ and serves as a solution of the following non-autonomous difference equation:

$$\begin{cases} x_{n+1} = h_{n+1}(x_n), \\ x_0 = x, \end{cases}$$

and is also represented by $o(x,\mathcal{H}) = \{x, h_1(x), h_1^2(x),\ldots, h_1^n(x),\ldots\}$. The autonomous discrete dynamic system is a specific case of the above non-autonomous difference equation when $h_n = h$ for all $n\geq 1$.

Definition 3.3 [19]. For $\varepsilon > 0$, the ε-open ball of (T,d) centred at $x\in T$ is represented by $B(x,\varepsilon) = \{y\in T: d(x,y) < \varepsilon\}$.

Definition 3.4 [19]. Consider a NADDS $(T,\mathcal{H})$, where (T,d) is a general metric space. Let $x\in T$ be a weak stable point of $\mathcal{H}$, denoted as $x \in ws(\mathcal{H})$, or $\mathcal{H}$ is weak stable at x. This means that for any $\varepsilon > 0$, there exists $\delta > 0$ and an integer I_x such that $o(z,\mathcal{H})\subset N_\varepsilon(\{h_1^j(z): j=0,...,I_x\})$ for every $z\in T$ with $d(z,x) < \delta$. $\mathcal{H}$ is weak stable if it is weak stable at every point $x\in T$. If the integer I_x (given the arbitrary $\varepsilon > 0$) can be chosen independent of each $x \in T$, the NADDS is known as uniformly weak stable.

Remark 3.1. The ε-ball around a nonempty subset A in X is defined as the set

$$N_\varepsilon(A)=\{x\in X: d(x,A)<\varepsilon\},$$

where $d(x,A) = \inf_{a\in A} d(x,A)$.

Definition 3.5 [19]. A NADDS $(T,\mathcal{H})$ is known as:

(a) Topologically Transitive: It is topologically transitive if, for each pair of open sets $U,V\subseteq T$, there exists $n\in\mathbb{N}_0$ such that $h_1^n(U)\cap V\neq\varnothing$. In this scenario, there exists a dense G_δ subset $Y\subset T$ such that $o(y,\mathcal{H})$ is dense in T. Each such point y with a dense orbit is referred to as a transitive point for $\mathcal{H}$, and $trans(\mathcal{H})$ is denoted by the corresponding collection of transitive points; or,

(b) Chain Transitive: It is chain transitive if for any $x,y\in T$, there is a δ-chain (see Definition 3.7) from x to y under $\mathcal{H}$.

Remark 3.2. [37] A subset of a topological space X is a G_δ set if and only if it is a countable intersection of open sets.

Definition 3.6 [19]. Set $I(T)$ as the collection of all non-empty compact subsets of T. The ε-open ball around a nonempty subset C in T is represented as $N_\varepsilon(C) = \{x\in T; d(x,C) <\varepsilon\}$, where $d(x,C) = \inf_{c\in C} d(x,c)$.

Let $\mathcal{H} = \{h_n\}_{n=1}^{\infty}$ be a sequence of continuous mappings $h_n: T\to T$, acting on a compact metric space (T,d). Initially, the stability of NADDS $(T,\mathcal{H})$ is explored, followed by the acquisition of a useful property related to weak stable points.

Theorem 3.1 [19]. *Let $(T,\mathcal{H})$ be a non-autonomous dynamical system (NADS) on a compact metric space (T,d). Then $x\in T$ is a weak stable point for $\mathcal{H}$ if and only if*

$$(wsp) \qquad \mathrm{cl}\big(o(x,\mathcal{H})\big) = \bigcap_{\delta>0} \mathrm{cl}\left(\bigcup_{J=0}^{\infty} h_1^J\big(B(x,\delta)\big)\right).$$

Proof. Let $(T,\mathcal{H})$ be a NADS on a compact metric space (T,d). Firstly, prove the backward direction (sufficiency) of the statement above. Suppose that $x\in T$ satisfies (wsp). For any $\varepsilon > 0$, $\delta_1 > 0$ is obtained such that $\mathrm{cl}(\bigcup_{J=0}^{\infty} h_1^J(B(x,\delta_1))) \subseteq N_{\frac{\varepsilon}{3}}(o(T,\mathcal{H}))$, by (wsp).

Set $\{U_i\}_{i=1}^{m}$ as a finite cover of T with open balls of radius $\frac{\varepsilon}{3}$. Set $G_x\subset\{1,\ldots,m\}$ such that $o(x,H)\subset\bigcup_{j\in Gx} U_j$, and $o(x,\mathcal{H})\cap U_j\neq\varnothing, \forall j\in G_x$.

For any $j\in G_x$, there exists $n_j\in\mathbb{N}_0$ such that $h_1^{nj}(x)\in U_j$. Set $I_x = \max\{n_j: j\in G_x\}$ and $0 < \delta < \delta_1$ such that if $z\in B(x,\delta)$ and $L\in\{0,\ldots,I_x\}$, then $d(h_1^L(z),h_1^L(x)) < \frac{\varepsilon}{3}$.

Given any $z\in B(x,\delta)$ and $v\in\mathbb{N}$, there exists $w\in\mathbb{N}_0$ such that $d(h_1^v(z), h_1^w(x))<\frac{\varepsilon}{3}$. If $j\in G_x$ with $h_1^w(x)\in U_j$, and $L\in\{0,\ldots,I_x\}$ with $h_1^L(x)\in U_j$, then by triangle inequality $d(h_1^v(z),h_1^L(z)) < d(h_1^v(z), h_1^w(x)) + d(h_1^w(x),h_1^L(x)) + d(h_1^L(x), h_1^L(z)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

This shows that $o(z,H) \subset N_\varepsilon(\{h_1^L(z): L = 0,\ldots,I_x\})$. Hence, by Definition 3.4, the weak stability of x, or, $x\in T$ is established as a weak stable point for $\mathcal{H}$, as $z\in B$(x,δ, as is arbitrary.

Secondly, it needs to be proven that the forward direction (necessity) of the logic in the statement above is true. Let $x \in ws(\mathcal{H})$, and choose any $y \in \bigcap_{\delta>0}^{J} cl(\bigcup_{J=0} h_1(B(x,\delta))$ and $\varepsilon_1 > 0$. By Definition 3.4, there exists $\delta_x > 0$ and $I_x \in \mathbb{N}$ related to the weak stability of x for $\varepsilon = \frac{\varepsilon_1}{3}$. Assume that there is a sequence $\{z_j\}_{j=0}^{\infty}$ in $B(x,\delta_x)$ in converging to x such that $y \in N_\varepsilon(o(z_j,\mathcal{H}))$ for every $j \in \mathbb{N}$. Hence, $y \in N_\varepsilon(o(z_j^L,\mathcal{H})) \subset N_{2\varepsilon}(\{h_1(z)$: $L=0,\ldots,I_x\})$. If the limits are taken when $j \to \infty$, and by continuity, it would yield $y \in cl(N_{2\varepsilon}(\{h_1(z);L=0,\ldots,I_x\})) \subset N_\varepsilon(o(x,\mathcal{H}))$. This shows that $y \in cl(o(x,\mathcal{H}))$, since ε_1 was arbitrary. Therefore, $cl(o(x,\mathcal{H})) = \bigcap_{\delta>0} cl(\bigcup_{J=0} h_1(B(x,\delta)))$.

Shadowing Property of NADDS

Definition 3.7 [19]. For $\delta > 0$, a δ-pseudo-orbit of $\mathcal{H}$ is a sequence $\{x_j\}_{j=0}^{\infty}$ in T such that $d(h_{j+1}(x_j), x_{j+1}) < \delta$ for $j \in \mathbb{N}_0$, where $\mathbb{N}_0$ represents the set of all nonnegative integers. A finite δ-pseudo-orbit $\{x_i\}_{i=0}^{b}$ is known as a δ-chain from x_0 to x_b with length $b + 1$.

Definition 3.8 [19]. For $\varepsilon > 0$, a sequence $\{x_i\}_{j=0}^{\infty}$ in T is known as ε-shadowed by a true orbit starting at a point $y \in T$ if $d(h_1(y),x_j) < \varepsilon$ if for all $j \in \mathbb{N}_0$.

Definition 3.9 [19]. A sequence of continuous maps $\mathcal{H}$ has the shadowing property if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that any δ-pseudo-orbit $\{x_j\}_{i=0}$ of $\mathcal{H}$ can be ε-shadowed by an orbit starting at $y \in T$, i.e., $d(h_1(y), x_j) < \varepsilon$ for all $j \in \mathbb{N}_0$.

The connection between weak stability and the shadowing property of $\mathcal{H}$ and $\mathcal{H}_p$ has been investigated. First, the following lemmas and propositions are needed.

Lemma 3.1 [19]. *Suppose that $\mathcal{H}$ has the shadowing property, then $\mathcal{H}_p$ has the shadowing property.*

Proof. Given $\varepsilon > 0$, $\delta > 0$ and suppose that $\{y_j\}_{j=0}^{\infty}$ is a δ-pseudo-orbit of $\mathcal{H}_p$. Hence, the following sequence

$$\{y_j\}_{j=0}^{\infty} = \{y_0, h_1(y_0), \ldots, h_1^{p-1}(y_0), y_1, h_{p+1}(y_1), h_{p+1}^{p+2}(y_1), \ldots, h_{p+1}^{2p-1}(y_1), y_2, h_{2p+1}(y_2),$$

$$h_{2p+1}^{2p+2}(y_2), \ldots, h_{2p+1}^{3p-1}(y_2), y_3, \ldots\}$$

is a δ-pseudo-orbit of $\mathcal{H}$. Consequently, there is $w \in T$ such that $d(h_1^j(w), y_j) < \varepsilon$ for $j \geq 0$, since $\mathcal{H}$ has the shadowing property. As a result, $\mathcal{H}_p$ has the shadowing property, due to $d(h_1^{p(j+1)}(w),y_{j+1}) < \varepsilon$.

Next, the following lemma will be proven:

Lemma 3.2 [19]. *Assume that $\mathcal{H}_p$ is chain transitive for $p \in \mathbb{N}$ and $\mathcal{H}$ has the shadowing property. Then, $\mathcal{H}_p$ is topological transitive.*

Proof. By applying Lemma 3.1, $\mathcal{H}_p$ inherits the shadowing property. Suppose that $B(u,r_1)$ and $B(v,r_2)$ are open balls of $u,v \in T$, respectively. According to Definition 3.9, for $0 < \varepsilon \min\{r_1,r_2\}$, there exists $\delta > 0$ such that every δ-pseudo-orbit of $\mathcal{H}_p$ can be ε-shadowed by some point $w \in T$. Definition 3.5 ensures the existence of a finite δ-chain $\{u = y_0,\ldots,y_n = v\}$ from u to v due to the chain transitivity of $\mathcal{H}_p$. Reapplying Definition 3.9, $d(w,u) < \varepsilon$ and $d(h_1^{pn}(w),v) < \varepsilon$. Consequently, $h_1^{pn}(B(u,r_1)) \cap (B(u,r_2)) \neq \emptyset$, and as per Definition 3.5, $\mathcal{H}_p$ is topological transitive.

One more proposition is needed.

Proposition 3.1 [19]. *Consider $(T,\mathcal{H})$ as a NADDS on a compact metric space T. Suppose that $(T,\mathcal{H})$ is topologically transitive. Then, $trans(\mathcal{H}) = ws(\mathcal{H})$.*

Proof. Suppose that $(T,\mathcal{H})$ is topologically transitive. According to Definition 5, there exists a dense G_δ subset $B(x,\delta)=Y \subset T$ such that $o(y,F)$ is dense in T, that is for every point x in T, any neighbourhood of x contains at least one point from Y (i.e., Y has no empty intersection with any nonempty open subset of T). Each such point with a dense orbit is a transitive point for $\mathcal{H}$. This implies that $\bigcap_{\delta>0} cl\big(\bigcup_{J=0}^{\infty} h_1^J(B(x,\delta))\big) = T, \forall x \in T$ (Note: $B(x,\delta)$ is a dense subset of T). Hence, by Theorem 3.1, $\bigcap_{\delta>0} cl\big(\bigcup_{J=0}^{\infty} h_1^J(B(x,\delta))\big) = ws(\mathcal{H})$, and as per Definition 3.5, $trans(\mathcal{H}) = T$. Hence, it can be concluded that $trans(\mathcal{H}) = ws(\mathcal{H})$.

Theorem 3.2 [19]. *Assume $\mathcal{H}_p$ is chain transitive for $p \in \mathbb{N}$. If $\mathcal{H}$ has the shadowing property, then $trans(\mathcal{H}) = ws(\mathcal{H})$.*

Proof. Let $\mathcal{H}_p$ be chain transitive for $p \in \mathbb{N}$, and suppose $\mathcal{H}$ has the shadowing property. According to Lemma 3.2, $\mathcal{H}_p$ is topological transitive. Therefore, by Proposition 3.1, $trans(\mathcal{H}) = ws(\mathcal{H})$.

NADDS AND THEIR INDUCED SYSTEMS

This section will explore the relationships between NADDS and their induced systems.

Definition 4.1 [19]. The Hausdorff separation $q(C,D)$ of $C,D \in I(T)$ is represented by $q(C,D) = \inf \{\varepsilon > 0: C \subseteq N_\varepsilon(D)\}$.

Definition 4.2 [19]. The Hausdorff metric on $I(T)$ is defined as $G_d(C,D) = \max \{q(C,D), q(D,C)\}$. $I_H(T)$. The collection of all finite subsets of T is dense in $I(T)$.

Note that, $(I(T), G_d)$ is compact if and only if (T,d) is compact.

Definition 4.3 [19]. Let $h:T\to T$ be a continuous map. Set $\bar{h}(B) = \{h(b): b \in B\}$ for every $B \in I(T)$. This defines a continuous mapping on $I(T)$. The induced system related to h is denoted as $(I(T),\bar{h})$.

The connection between the weak stability of (T,H) and its induced system $(I(T),\bar{H})$ has also been inspected. The aim is to prove the following theorem.

Theorem 4.1 [19]. *Let (T,H) be a NADDS, and $(I(T),\bar{H})$ be its induced system. Then,H is weak stable if and only if $\bar{H}$ is weak stable.*

The following proposition is needed to prove this theorem.

Proposition 4.1 [19]. *Let (T,H) be a NADDS on a compact metric space (T,d). Suppose that (T,H) is weak stable; then, it is uniformly weak stable.*

Proof. Given $x \in T$ and any $\varepsilon > 0$. Since (T,H) is weak stable, by Definition 3.4, there exists $\delta_x > 0$ and $I_x \in \mathbb{N}$ such that $o(z,H) \subset N_\varepsilon(\{h_1^J(z): J = 0,\ldots,I_x\})$ for any $z \in T$ with $d(z,x) < \delta_x$. Set a finite collection $\{x_i \in T; i = 1,\ldots,k\}$ such that $T \subseteq \bigcup_{i=0}^{k} B(x_i,\delta_{xi})$. If $I = \max\{I_{xi}; i=1,\ldots,k\}$ is set and since ε was arbitrary, then the uniform weak stability of the system is obtained.

Moving on to the proof of Theorem 4.1,

Proof of Theorem 4.1. Let $\varepsilon_1 > 0$, and suppose that (T,H) is weak stable. By Proposition 4.1, then (T,H) is uniformly weak stable. Hence, according to Definition 3.4, let $I \in \mathbb{N}$ associated with $\varepsilon = \frac{\varepsilon_1}{3}$. For any $U \in I(T)$, it yields

$$o(U,\bar{H}) = \bigcup_{x\in U} o(x,\bar{H}) \subset \bigcup_{x\in U} N_\varepsilon\big(\{h_1^j(x) : j = 0,\ldots,I\}\big) = N_\varepsilon\left(\left\{\overline{h_1^j}(U) : j = 0,\ldots,I\right\}\right),$$

This shows that $(I(T),\bar{H})$ is weak stable.

$G_d(\{a\},\{b\})$ $_H(T)$. It's important to note that as per

$= \max\{q(\{a\},\{b\}), q(\{b\},\{a\})\}$

$= \max\{\inf\{\varepsilon > 0\,; \{a\} \subseteq N_\varepsilon(\{b\})\}, \inf\{\varepsilon > 0\,; \{b\} \subseteq N_\varepsilon(\{a\})\}\}$

$$= \max\begin{Bmatrix} \inf\{0 < d(a,\{b\}) = \inf_{b\in\{b\}} d(a,b) < \varepsilon\,; \{a\} \subseteq N_\varepsilon(\{b\})\}, \\ \inf\{0 < d(b,\{a\}) = \inf_{a\in\{a\}} d(b,a) < \varepsilon\,; \{b\} \subseteq N_\varepsilon(\{a\})\} \end{Bmatrix}$$

$= \max\{\inf\{d(a,b)\}, \inf\{d(b,a)\}\}$

$= \max\{d(a,b), d(b,a)\}$

$= d(a,b) \quad \forall a,b \in T,$

$$G_d\left(\overline{h_1^n}\{a\}, \overline{h_1^n}\{b\}\right) = d\left(\overline{h_1^n}\{a\}, \overline{h_1^n}\{b\}\right), \quad \forall a,b \in T.$$

and also

Since $(I(T),H)$ is weak stable, by Definition 3.4, and the proof above, (T,H) is weak stable.

CONCLUSION

In summary, the identified condition ensures the existence of weak stability in a NADDS, streamlining the identification of weak stable point efficiently. Furthermore, a connection between weak stability and the shadowing property has been established, facilitating the identification of weak stable point efficiently in systems exhibiting shadowing. By identifying the transitive point of that system, then they automatically will be the weak stable point. Finally, the connection between the weak stability of a NADDS and its induced system has been established. This connection proves valuable when examining specific subsets of a compact metric space. It ensures that after selecting these subsets, the induced systems also maintain weak stability if the original NADS possesses this characteristic. This streamlined approach reduces the need to reassess the weak stability of the induced systems.

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

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