PENERBIT UMT
Menjana Khazanah Ilmuwan

Journal of Mathematical Sciences and Informatics
Journal Homepage: https://journal.umt.edu.my/index.php/jmsi
eISSN: 2948-3697
DOI: http://doi.org/10.46754/jmsi.2023.12.004

JMSI Journal of Mathematical Sciences and Informatics

ON THE APPROXIMATION OF THE FUNCTION ON THE UNIT SPHERE BY THE SPHERICAL HARMONICS

ABDUMALIK RAKHIMOV*

Faculty of Engineering, International Islamic University Malaysia; abdumalik@iium.edu.my

**Corresponding author: abdumalik@iium.edu.my*

ARTICLE INFO

Article History:
Received 13 JULY 2023
Accepted 22 SEPTEMBER 2023
Available online
25 JANUARY 2024

Section Editor:
Roslan Hasni

Keywords:
Approximation;
Fourier series on a sphere;
Summability;
Functions spaces;
Distributions

ABSTRACT

In this paper, the convergence and summability of the Fourier series for distributions in domains coinciding with smooth functions in eigenfunction expansions of the Laplace operator on the unite sphere are explored. The representation of distributions defined on the unit sphere by their Fourier-Laplace series, employing spherical harmonics in different topologies, is examined. The main focus lies in studying the Cesaro method for summing such a series.

2020 Mathematics Subject Classification: 41A30, 42B35, $2C10

INTRODUCTION

The approximation of a function defined on a unit sphere with an orthogonal function system is important in solving boundary and initial problems for mathematical physics equations, such as heat or wave equations, within a unit ball. An example of such an orthogonal function system is the set of orthogonal polynomials on a unit sphere.

Spherical harmonics are the restrictions of space homogeneous harmonic polynomials on a unit sphere S^N, $N \geq 3$. If Y_n and Y_m represent spherical harmonics degree n and m, $n \neq m$, respectively, then they are orthogonality in $L_2(S^N)$:

$$\int_{S^N} Y_n(x)\, Y_m(x) d\sigma(x) = 0, \ \mathrm{n} \neq \mathrm{m}$$

Here $d\sigma(x)$ denotes the surface element on a unit sphere.

The set of all spherical harmonics of degree n, denoted as $a_n = \frac{(N+n)!}{(N!n)!} - \frac{(N+n\text{-}2)!}{N!(N\text{-}2)!}$, forms a dimensional subspace of $L_2(S^N)$. Let $\{Y_1^n, Y_2^n, ..., Y_{a_n}^n\}$ represent the orthonormal basis of this subspace. The union of all such bases will constitute an orthonormal basis in $L_2(S^N)$.

Denote by $P_n^{\lambda}(t)$ the Gegenbauer polynomials (refer to [6]). These polynomials can be generated from the function $(1\text{-}2th+h^2)^{-\lambda}$ as the coefficients in the expansion of this generating function in the Maclaurin series with respect to the variable h. Then the following summation formula is valid:

$$P_n^{\frac{N-2}{2}}(cos\gamma) = \frac{2\,\pi^{\frac{N}{2}}}{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}\sum_{j-1}^{a_n} Y_j^n(x)Y_j^n(y),$$

where γ, $0 \leq \gamma \leq \pi$, is the spherical distance between x and y.

Let $f \in L_2(S^N)$. The Fourier coefficients with respect to the orthonormal system $\{Y_1^n, Y_2^n, \ldots, Y_{a_n}^n\}$ by:

$$f_j^n = \int_{S^N} f(x)Y_j^n(x)d\sigma(x), \quad j = 1, 2, 3, \ldots, a_n$$

The Fourier-Laplace series of the function f is defined as follows:

$$f(x) = \sum_{n=1}^{\infty} Y_n^{\frac{N-2}{2}}(f, x) \tag{1}$$

where

$$Y_n^{\frac{N-2}{2}}(f, x) = \sum_{j=1}^{a_n} Y_j^n(x) f_j^n$$

Taking into account the summation formula above for the Gegenbauer polynomials,

$$Y_n^{\frac{N-2}{2}}(f, x) = \frac{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{2\,\pi^{\frac{N}{2}}}\int_{S^N} f(y)P_n^{\frac{N-2}{2}}(cos\gamma)\ \ d\sigma(y), \tag{2}$$

Note that for any function $f(y)$ from $L_2(S^N)$, the following energetic equation, known as Parseval's equality, is valid:

$$||f|| = \sqrt{\sum_{n=1}^{\infty} \left\| Y_n^{\frac{N-2}{2}}(f, x) \right\|^2}$$

Here, $||f||$ represents the norm of the function f in $L_2(S^N)$.

Hence, from the Parseval's equality, it follows that for any function f in $L_2(S^N)$, its Fourier-Laplace series (1) converges in the norm of the space $L_2(S^N)$.

Numerous research works and literature are dedicated to the study of the Fourier-Laplace series on a sphere (1). Notably, in the book [6], the author analyses problems of summability, localisation problems, and almost everywhere convergence. In [6], these problems are investigated in the Lebesgue spaces $L_p(S^N)$, $p \geq 1$. Additionally, in [1], the author develes into problems of uniform convergence and localisation of the Cesaro means of the series (1) within the Nikolskii classes. This problem is also explored in [2] for the singular functionals, while the uniform convergence of these series studied in [3]. The convergence of the Cesaro means of the Fourier-Laplace Series is examined in [4]. Furthermore, problems within the spaces of distributions are investigated in [5], [6], [7]. For a foundational understanding of the Four-Laplace series, one can refer to the book [8].

THE CESARO MEANS OF THE FOURIER-LAPLACE SERIES

Let α be a non-negative number. Then, the Cesaro means of order α of the partial sums of the Fourier-Laplace series (1) are defined as follows:

$$S_n^\alpha f(x) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha Y_k^{\frac{N-2}{2}}(f, x) \tag{3}$$

Here, $A_m^\alpha = \frac{\Gamma(\alpha+m+1)}{\Gamma(\alpha+1)m!}$. It is clear that if $\alpha= 0$, then (3) is the partial sum of (1).

Substituting (2) into (3) and changing the order of integration and summation yields:

$$S_n^\alpha f(x) = \int_{S^N} f(y)\Theta^\alpha(x, y, n) \quad d\sigma(y) \tag{4}$$

where

$$\Theta^\alpha(x,y,n) = = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha \frac{\left(n+\frac{N-2}{2}\right)\Gamma\left(\frac{N-2}{2}\right)}{2\pi^{\frac{N}{2}}} P_n^{\frac{N-2}{2}}(cos\gamma). \tag{5}$$

Thus, the Cesaro means of the partial sums of the Fourier-Laplace series (3) can be considered as an integral operator (4) with the kernel (5).

For the kernel (5), the following estimations are valid [1]:

Theorem 1. If $\alpha > -1$ and $\left|\frac{\pi}{2} - \gamma\right| \le \frac{n}{n+1} \cdot \frac{\pi}{2}$, then for $n \to \infty$

$$\Theta^\alpha(x,y,n) = O(1) \cdot \frac{n^{\frac{N-1}{2}-\alpha}}{(2\sin\gamma)^{\frac{N-1}{2}} \cdot \left(2\sin\frac{\gamma}{2}\right)^{1+\alpha}} + \\ + \frac{O\left(n^{\frac{N-1}{2}-\alpha-1}\right)}{(\sin\gamma)^{\frac{N+1}{2}}\left(\sin\frac{\gamma}{2}\right)^{1+\alpha}} + \frac{O\left(\frac{1}{n}\right)}{\left(\sin\frac{\gamma}{2}\right)^{1+N}},$$

if $\alpha > -1$ and $0 < \gamma_0 \le \gamma \le \pi$, then for $n > 1$

$$\Theta^\alpha(x,y,n) = O(1) \cdot n^{N-1-\alpha}$$

if $\alpha > -1$ and $0 \le \gamma \le \pi$, then for $n > 1$

$$\Theta^\alpha(x,y,n) = O(1) \cdot n^N$$

THE SPACES OF THE DISTRIBUTIONS ON S^N.

Denote by $C^\infty(S^N)$ a space of infinitely differentiable functions on a sphere S^N. This is a linear vector space and can be equipped with the topology of the uniform convergence of the sequence of functions and their derivative of any order. Denote this topological space as $\varepsilon(S^N)$. Denote by $\varepsilon'(S^N)$ the space of the linear continuous functionals on $\varepsilon(S^N)$.

Note that a set of all polynomials $Y_j^n(x)$ belong to $\varepsilon(S^N)$. For any functional $f \in \varepsilon'(S^N)$, its Fourier coefficients are defined as the action of this functional on the function $Y_j^n(x)$ as follows:

$f_j^n = < f, Y_j^n(x) >$. The corresponding decomposition of the functional f using the orthogonal system of test functions is represented as follows:

$$f = \sum_{n=1}^{\infty} \sum_{j=1}^{a_n} Y_j^n(x) f_j^n \,.$$

This decomposition always converges in the weak topology, which is the topology of the space $\varepsilon'(S^N)$.

MAIN THEOREM

In addition to the the weak (in $\varepsilon'(S^N)$) and strong (in $L_2(S^N)$) topologies mentioned above, one can consider other topologies for the equality above in the sub domains of S^N, where a distribution coincides with the regular function. For example, if a singular distribution coincides with a continuous function in $V \subseteq S^N$, then the examination of uniform convergence or summability of the series in V is possible.

For the classifications of the singularities of the distributions, Sobolev spaces $W_2^\ell(S^N)$ are employed, where ℓ any real number. Then, $\cap W_2^\ell(S^N) = \varepsilon'(S^N)$

The main result of this paper is encapsulated in the following theorem.

Theorem 2. *Let in $V \subseteq S^N$ be a domain and f be a continuous functional in V from the space $W_2^\ell(S^N)$ $\cap \varepsilon'(S^N)$, $\ell > 1/2$. Then the series (4) uniformly converges in any compact set from V if $\alpha \geq \frac{N-1}{2} + \ell$.*

PROOF OF THEOREM 2

Introduce a new function $g_1(x)$ that is continuous on the sphere S^N and coincides with the functional f in V. Set $g_2 = f - g_1(x)$. The uniform convergence of $S_n^\alpha g_1(x)$ in any compact set from V to f follows from estimations of the kernel (5) in Theorem 1 and the following theorem [1].

Theorem 3. Suppose $g(x)$ is a continuous function from the Nikolskíi class $H_2^\ell(S^N)$, and $\alpha + \ell \geq \frac{N-1}{2}$, $\ell > 1/2$. Then

$$\lim_{n\to\infty} S_n^\alpha g(x) = g(x),$$

uniformly in on S^N.

Note that the functional g_2 is equal to zero in the domain V. Thus, to prove Theorem 2, it needs to be established that

$$\lim_{n\to\infty} S_n^\alpha g_2(x) = 0,$$

This follows from the following theorem on the localisation [5].

Theorem 4. Suppose $g(x)$ is a functional from the class $H_2^\ell(S^N) \cap \varepsilon'(S^N)$, and $\alpha \geq \ell + \frac{N-1}{2}$. Then

$$\lim_{n\to\infty} S_n^\alpha g(x) = 0,$$

uniformly in any compact set from $S^N \backslash supp\,(g)$.

Theorem 2 is thus proved.

CONFLICTS OF INTEREST

The author declares that there are no conflicts of interest.

ACKNOWLEDGEMENT

The authors would like to thank the International Islamic University of Malaysia for granting research leave.

REFERENCES

[1] Pulatov, A. K. (1981). On the uniformly convergence and localization of arithmetic means of Fourier-Laplace series. *Journals Soviet Doclads*, *258*(3), 554-556.

[2] Rakhimov, A. A. (2023). Summabilty of the Fourier-Laplace series in the Nikol'skii spaces. *AIP Conference Proceedings - 6th International Conference on Mathematical Applications in Engineering, 2880*, 040002, https://doi.org/10.1063/5.0166088

[3] Rakhimov, A. A. (2016). On the uniform convergence of Fourier series. *Malaysian Journal of Mathematical Sciences*, *10*(2016), 55–60.

[4] Rakhimov, A. A. (2023). On the convergence of the Cesaro Mean of the Fourier-Laplace series. *Journal of Computer Science & Computational Mathematics*, *13*(2), 47-49. https://www.jcscm.net/cms/?action=showissue&volume=13&issue=2

[5] Rakhimov, A. A. (2017). On the uniform convergence of Fourier series on a closed domain. *Eurasian Mathematical Journal*, *8*(3), 60–69.

[6] Rakhimov, A. A. (2019). Localization of the spectral expansions associated with the partial differential operators. In Taş, K., Baleanu, D., & Machado, J. (Eds.), *Mathematical Methods in Engineering* (pp. 217-233). Cham: Springer.

[7] Rakhimov, A. A. (2009). Some problems of summability of spectral expansions connected with Laplace operator on sphere. *ArXiv. Math*, *SP*(2009), 1-8. https://doi.org/10.48550/arXiv.0902.4868

[8] Topuriya, S. B. (1987). *The Fourier-Laplace series on a sphere*. Tbilisi University Press.