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FUNCTORIAL PROPERTIES OF THE SPACE *Z*(*X*) FOR STRATIFIABLE SPACES

ZHURAEV TURSUNBOY[1*], ZABIDIN SALLEH[2], NURILLAEV MUZAFFAR[3]

[1]Faculty of Physics and Mathematics, Tashkent State Pedagogical University named after Nizami, Tashkent, Uzbekistan; tursunzhuraev@mail.ru. [2]Special Interest Group on Modelling and Data Analytics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia; zabidin@umt.edu.my. [3]Faculty of Physics and Mathematics, Tashkent State Pedagogical University named after Nizami, Tashkent, Uzbekistan; nurillaev_muzaffar@mail.ru

**Corresponding author: tursunzhuraev@mail.ru*

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ABSTRACT

In the realm of stratifiable spaces and their continuous mappings into themselves, a construction ¢ within the domain of Coty, known as the test space, defines a covariant functor within this category. This construction Z defines a functor $Z{:}S\rightarrow S$, which allows each stratifiable space X to be immersed in a closed manner into some other space $Z(X)$, which is a stratifiable space with "good" functorial, geometric and topological properties. It is shown that the functor $Z{:}S'\rightarrow S'$ is a normal, open and monadic functor in this category S of stratifiable spaces and continuous mappings into itself. Furthermore, an exploration of the dimensional properties of the space $Z(X)$ is conducted for the stratifiable space X, defined for each $n\in N_+$ subfunctor $Z_n{:}S\rightarrow S$ of the functor Z, for which the dimension satisfies the inequality: $\dim Z_n(X) \leq n \dim X+n$-1.

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INTRODUCTION

A simplicial complex is defined as a set K for which its representation is fixed in the form of a union of some family of closed simplices, each two of which are not intersected, either one of them is a face of the other, or is intercepted by a simplex that is a face of each of them. The set K itself is called the body of the complex under consideration. In the presence of a complex, its body is uniquely determined, but one and the same set can be the body of various complexes. A simplicial complex is called complete if the complex K by an arbitrary given simplex $\sigma\in K$ also contains all the faces of this simplex.

Let K be a simplicial complex. By $K^{(n)}$, we denote the set of n-dimensional simplices (skeletons) of the simplicial complex K. $|K|$ is a polyhedron, i.e.$|K| = \cup K$ in weak topology.

For each simplex $\sigma\in K$, the barycentre, the interior and the boundary of the simplex σ are denoted by $\hat{\sigma}$, $\mathring{\sigma}$ and $\partial\sigma$, respectively. A simplex, whose vertices are $\vartheta_1, \vartheta_2,\ldots,\vartheta n$, is denoted by $< \vartheta_0, \vartheta_1, K, \vartheta n>$. Thus, $\tau \leq \sigma$ ($\tau < \sigma$) implies that τ is an integral part (facade or component) of the simplex

σ. For each point $x\in|K|$ through $(x(\tilde{\sigma})$ of the simplex $\sigma\in K$, the barycentric coordinates with respect to the barycentre of the S_dK subdivision are denoted.

A simplicial subdivision of a simplex $\sigma\in K$ is a family $P = \{\sigma_i : i = \overline{1,k}\}$ of simplices in K satisfying the following three conditions:

a) $\sigma = \cup_{i=1}^{k} \sigma_i$;

b) For any $i, j \le k$, the intersection $S_i \cap S_j$ is either empty or is a common face of the simplices σ_i and σ_j.

c) For $i = 1,2,...,k$, all faces of the simplex σ_i belong to P.

The fineness of the simplicial subdivision $\{\sigma_i : i = 1,k\}$ of the simplex σ is defined as the largest among the numbers $\delta(\sigma_1), \delta(\sigma_2),...,\delta(\sigma_k)$. For each simplex $\sigma = \overline{a_0, a_1,...,a_m}$, the point

$$b(\sigma) = \frac{1}{m+1}a_0 + \frac{1}{m+1}a_1 + \ldots + \frac{1}{m+1}a_m$$

is referred to as the barycentre (or centre of gravity) of the simplex σ. It is evident that $b(\sigma)\in\sigma$, and $b(\sigma)$ do not belong to any $(m$ -1)-dimensional face of the simplex σ.

Consider an arbitrary simplex $\sigma = < a_0, a_1,...,a_m >$. For each decreasing sequence $\sigma_0 \supset \sigma_1 \supset ... \supset \sigma_k$ of different faces of the simplex σ, the points $b(\sigma_0), b(\sigma_1),..., b(\sigma_k)$ are linearly independent. The family Π of all simplices of the form $< b(\sigma_0), b(\sigma_1),..., b(\sigma_k) >$ will be a simplicial subdivision of the simplex σ. Each $(m$-1)-dimensional simplex $T\in \Pi$ is a face of one or two m-dimensional simplices of the family Π, depending on whether the simplex is contained or not T in some $(m$-1)-dimensional face of the simplex σ. This particular simplicial subdivision, as described above, is referred to as the barycentric subdivision of the simplex σ.

Consider a number $t\in [0,1]$ and $\sigma(t) = \{x\in\sigma: 0 \le \hat{x}(\sigma) \le t, (\hat{x}(\sigma))$ are barycentric coordinates of the point $x\}$. For each $t\in[0,1]$, the set $\sigma(t)$ is an open neighbourhood of the boundary $\partial\sigma$ of the simplex σ, and $\sigma(t) = cl_\sigma\, \sigma(t)$. Note that $\sigma(1) = \sigma \setminus \{\hat{\sigma}\}$.

Each point $x\in\sigma$ (1) can be expressed as as:

$$x = \left(1 - x(\hat{\sigma})\right)\, \pi_\sigma(x) + x(\hat{\sigma})\hat{\sigma},$$

where $\pi_\sigma(x)\in\partial\sigma$ and $\pi_\sigma(x)$: $\sigma(1) \to \partial\sigma$ is a radial projection for each $\sigma\in K$. Furthermore, each point $x\in \upsilon_0,\upsilon_1,...,\upsilon_n$ can be written as:

$$x = \sum_{i=0}^{n} b(\upsilon_i)\upsilon_i,$$

where $\upsilon_i\in K^0$, and $b(\upsilon_i)$ represent the barycentric coordinates of the point x.

MAIN PART

Let X be a topological space, and $|F(X)|$ be a complete simplicial complex whose vertices are points of X, i.e. $|F(X)^0|= X$. The space $|F(X)|$ is equipped with the weak topology. Now, a topology on the space $|F(X)|$ is defined, denoting the basis of open sets as $Z(X)$, which consists of W open in $F(X)$, and satisfy the following conditions:

o1. $W \cap X$ open in X;

o2. $|F(W \cap X)| \subset W$;

i.e. $\tau_{Z(X)} = \{W\in\tau_{|F(X)|}: W$ satisfies the conditions o1-o2$\}$.

Condition o2 means that every simplex $\sigma \in F(X)$ is contained in W if all vertices of this simplex σ lie entirely in $W \cap X$.

For the subset $A \subset X$, the set $F(A)$ is a subcomplex of the full complex $F(X)$ and $Z(X)$ is a subspace of the space $Z(X)$. Obviously, $Z(A)$ is closed in $Z(A)$ if is closed in X.

For each $n \in Z_+ = N \cup \{0\}$, define $Z_n(X) = |F(X)^n|$, which is a subspace of $Z(X)$. Then, $Z_0(X) \cong X$ and $Z(X) = \cup_{n=0}^{\infty} Z_n(X)$. It is evident that for any $n \in Z_+$, the subspace $Z_n(X)$ is closed in $Z(X)$.

Let us introduce the following notation:

$$T(A) = \{\sigma \in F(X) \setminus F(A) \colon \sigma \cap A \neq \varnothing\};$$

$$M(A) = \{x \in Z(X) \colon \text{there exists } \sigma \in F(A) \text{ such that } x(\hat{\sigma}) < 0\};$$

$$T_n(A) = T(A) \cap (F(X)^n \setminus F(X)^{n-1});$$

$$M_n(A) = Z(A) \cup (M(A) \cap Z_n(X));$$

For each $\varepsilon \in (0,1)^{T(A)}$ and for each $n \in N$, we define the set:

$$M(A,\varepsilon) = \cup_{n \in Z_+} M_n(A,\varepsilon),$$

where $M_0(A,\varepsilon) = Z(A) = |F(A)|$ and

$$M_n(A,\varepsilon) = Z(A) \cup \{\sigma(\varepsilon(\sigma)) \cap \pi_\sigma^{-1}(M_{n-1}(A,\varepsilon)) \colon \sigma \in T_n(A)\}.$$

Then the equality $M_n(A,\varepsilon) \cap X = A$ holds.

For each open set U of the space X, the set $M(U,\varepsilon)$ is open in $Z(X)$. In this case, the family $B(M) = \{(U,\varepsilon) \colon U \text{ is open in } \varepsilon \in (0,1)^{(U)}\}$ is an open base of the space $Z(X)$.

Therefore, if for every $n \in N$ and every $\varepsilon \in (0,1)^{T_1(A) \cup T_2(A) \cup \ldots \cup T_n(A)}$, the set $M_n(A,\varepsilon)$ is defined, then the family $B(M) = \{(M_1, (U,\varepsilon)\} \colon U$ is open in X and $\varepsilon \in (0,1)^{T_1(U)}\}$ is an open base for $Z_1(X)$, i.e the following holds.

Lemma [5]. *Families $\{M(U,\varepsilon) \colon U$ is open in X and $\varepsilon \in (0,1)^{T_1(U)}\}$ and $\{(M_1, (U,\varepsilon)\} \colon U$ is open in X and $\varepsilon \in (0,1)^{T_1(U)}\}$ is the base of the space $Z(X)$, (respectively, the space $Z_1(X)$).*

Let $A \subset X$, then the subspace $Z(X) \setminus Z(A)$ of the space $Z(X)$ consists of points of the form: $z = (1-t)x + ty$, where $x \in Z(X \setminus A)$, $y \in Z(A)$, $t \in [0,1)$, i.e.

$$Z(X) \setminus Z(A) = \{z = (1-t)x + ty \in Z(X) \colon x \in Z(X \setminus A), y \in Z(A), t \in [0,1)\}.$$

Obviously, if the set A is closed in X, then $Z(A)$ is closed in $Z(X)$. It should be noted that the set itself is also closed in $Z(X)$, i.e. X; $Z_1(X)$.

Note that for $z \in Z(X)$, based on the structure of the space $Z(X)$, the point has the form: $z = m_0 \bar{x}_0 + m_2 \bar{x}_2 + \ldots + m_n \bar{x}_n$, where $\sum_{i=0}^{n} m_i = 1$, $m_i \geq 0$, and $x_i \in X$. In other words, the point z belongs to the simplex $\sigma = <\bar{x}_0, \ldots, \bar{x}_n>$, whose vertices are points x_i of space X, denoted by $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n$, which are the points (vertices) of the simplex $\sigma = <\bar{x}_0, \ldots, \bar{x}_n>$.

If $z \in Z_1(X)$, then the point z has the form $z = x$, where $x \in X$. Hence, for each point $z \in Z(X) \setminus Z_1(X)$ there exists a simplex $\sigma \in Z(X)$ such that $z \in \sigma = <\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n>$ and $z = \sum_{i=0}^{n} m_i \bar{x}_i, m_i > 0$, and $\sum_{i=0}^{n} m_i = 1$. The vertices of the simplex $\sigma = <\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_n>$ are referred to supports of the point $z \in \sigma$, i.e. $\mathrm{supp} z = \{x_0, x_1, \ldots, x_n\}$.

In this case, there exists at least j_1 and $j_2 \in \overline{0, n}$ such that $m_{j1} \neq 0$ and $m_{j2} \neq 0$, indicating that the point $z \in Z(X) \backslash Z_1(X)$ has at least two points in its support. Therefore, if $z \in Z(X) \backslash Z_1(X)$, then $|\text{suppz}| \geq 2$. On the other hand, given the construction of the space $Z(X)$, for any $n \in N$, there is a point $z \in Z(X)$ such that $|\text{suppz}| \geq n$. If $z \in Z_n(X)$, then $|\text{suppz}| \geq n$.

In the paper [1] by R. Coty, it was demonstrated that the space $Z(X)$ is stratifiable (in short, a S-space) if and only if X is a S-space.

Let $f:X \to Y$ be a continuous mapping between the S-spaces X and Y. The continuous mapping $Z(f):Z(X) \to Z(Y)$ is defined as follows: for each point $z \in Z(X)$ and $z \in \langle \bar{x}_0, \ldots, \bar{x}_k \rangle$

$z_x = m_0 \bar{x}_0 + \ldots + m_k \bar{x}_k$, $\sum_{k=1}^{k} m_i = 1$, $m_i \geq 0$ assign point $z_y \in Z(Y)$, $z_y = \langle \bar{y}_0, \ldots, \bar{y}_k \rangle$ $z_y = m_0 \bar{y}_1 + m_2 \bar{y}_2 + \ldots + m_k \bar{y}_k$, where $f(\bar{x}_i) = \bar{y}_i$.

The mapping $Z(f):Z(X) \to Z(Y)$ is simplicial. Therefore, $Z(f)$ is continuous. Note that $Z(f)(Z(X)) = Z(f(X))$ and $Z(f)(Z(X) \backslash X) \subset Z(Y) \backslash Y$. Hence, if the mapping $f:X \to Y$ is surjective, then $Z(f):Z(X) \to Z(Y)$ is also surjective and vice versa.

a) If id_x: $X \to X$ is the identity mapping, then $Z(id_x)$: $Z(X) \to Z(X)$ is also the identity mapping, i.e. $Z(id_x) = id_{z(x)}$.

b) Let p: $X \to Y$ and g: $Y \to L$ be continuous mappings between the stratifiable spaces X, Y and L.

If $f = g \circ p$: $X \to L$, where the diagram

$$X \overset{p}{\to} Y \quad f]\,[g \quad L \tag{1}$$

is commutative, then $Z(f) = Z(p \circ g)$: $Z(X) \to Z(L)$, which is the diagram

$$Z(X) \overset{Z(p)}{\to} Z(Y) \quad Z(f)]\,[Z(g) \quad Z(L) \tag{2}$$

is also commutative. Hence, we see that the mapping $Z:X \to Z(X)$ forms a covariant functor between stratifiable spaces and their continuous mappings.

If $f = p \circ g$, then $Z(f) = Z(p \circ g) = Z(p) \circ Z(g)$. Hence, the mapping of the stratifiable space Z: $X \to Z(X)$ is functorial. Each mapping f: $X \to Y$ can be associated with the above defined simplicial mapping $Z(f):Z(X) \to Z(Y)$. From properties a) and b), we can assert that this mapping Z is a covariant functor on the category of S- spaces and continuous mappings into itself.

Thus, the following is true:

Theorem 1. *is a covariant functor.*

Note that the functor Z: $S \to S$ has the following properties:

1°. From the construction of the space $Z(X)$ and Lemma [5], it follows that the functor Z: $S \to S$ preserves the entire space $\omega(Z(X)) \leq \omega(X)$, i.e. $\omega(Z(X)) \leq \omega(X)$.

2°. For $X \in S$, the mapping or embedding η_z: $X \to Z(X)$ is constructed by defining $\eta(X) = \langle x \rangle$, where $x \in Z(X)$, and x is vertex, i.e. x is a zero-dimensional simplex in $Z(X)$. The mapping η: $X \to Z(X)$ is continuous and one-to-one, implying that η_z: $X \to Z(X)$ is an embedding.

An immediate check shows that the functor Z: $S \to S$ preserves

3°. a point and an empty set;

4°. intersection of closed sets;

5°. preimages of mappings;

6°. continuous in the sense of Shchepin [2];

7°. monomorphic; and,

8°. epimorphic.

Note that the functor Z: $S \to S$ with finite supports does not have finite degrees. Hence, the functor Z: $S \to S$ preserves a point, an empty set, intersections of closed sets, and inverse images of mappings. It is continuous, epimorphic, and monomorphic, i.e. the functor Z: $S \to S$ is normal.

For each $n \in N_+$ defube $Z_n(X) = \{z \in Z(X)\}$: $|\text{suppz}| \leq n\text{-}1\}$, where $z_x = \sum_{i=0}^{n} m_i \bar{x}_i$, $\text{suppz}_x = \{x_0, x_1, \ldots, x_n\}$, $\sum_{i=0}^{n} m_1 = 1$, $m_i \geq 0$, $\text{suppz}_x \subset X$.

a) The set $Z_n(X)$ is closed in $Z(X)$;

b) for $n = 0$, $Z_0(X) \cong X$;

c) for each mapping f: $X \to Y$ between S spaces, the mapping $Z_n(f)$:$Z_n(X) \to Z_n(Y)$ is defined as follows: $Z_n(f) = Z(f)|z_{n(x)}$: $Z_n(X) \to Z_n(Y)$.

The mapping $Z_n(f)$:$Z_n(X) \to Z_n(Y)$ is also continuous, i.e. Z_n: $S \to S$ is a subfunctor of the functor Z. The functor Z_n: $S \to S$ possesses all properties of the functor Z: $S \to S$. Functor Z_n: $S \to S$ is a functor with finite n-supports.

Since the mappings $Z(f)$ and $Z_n(f)$ are simplicial, we can assert:

Theorem 2. *A mapping f: $X \to Y$ between S-spaces is open if and only if $Z(f)$: $Z(X) \to Z(Y)$ is open (or $Z_n(f)$: $Z_n(X) \to Z_n(Y)$ is open).*

The following is easily proved.

Theorem 3. *A map f: $X \to Y$ between S-spaces mapping $Z(f)$: $Z(X) \to Z(Y)$, $Z_n(f)$: $Z_n(X) \to Z_n(Y)$ is a homeomorphism if and only if is a homeomorphism.*

Note that the covariant functors Z: $S \to S$ and Z_n: $S \to S$ is a normal functor.

A monad (or triplet) on the category G is a triplet $T = \langle F, \psi, \eta \rangle$, where F:$G \to G$ is a functor η: $Id \to F$ (unit) and ψ:$F^2 \to F$ (multiplication) are natural transformations. Additionally, for each object X, the following equalities are satisfied:

$$\psi_X \circ F(\eta_X) = id_{F(X)},$$

$$\psi_X \circ \eta_{F(X)} = id_{F(X)}$$

$$\psi_X \circ F(\psi_X) = \psi_X \circ \psi_{F(X)}$$

A functor F is called monadic if it can be included in some monad.

Theorem 4. *The covariant functor Z: $S \to S$ is monadic.*

Proof. As previously mentioned, the mapping η: $Z(X)\to Z(Y)$ which assigns to each point $x\in X$ the vertex $\overline{x}$ of the simplex in $Z(X)$ is continuous, i.e. $\eta(x)=\overline{x}$.

Let $z\in Z(Z(X))=Z^2(X)$. Then the point z has the form $z=\sum_{k=1}^{k}\mu_i z_i$, where $z_i\in Z(X)$,

$$z_i=\sum_{i=0}^{l_i} m_s^i\overline{x_s^i},\ \sum_{i=0}^{k}\mu_i=1,\ \sum_{s=0}^{l_i} m_s^i=1,\ x_s^i\in X,$$

$$z=\sum_{i=0}^{k}\mu_i z_i=\mu_0 z_0+\mu_1 z_1+\ldots+\mu_k z_k=\mu_0\sum_{s=0}^{l_0} m_s^0\overline{x_s^0}+\mu_1\sum_{s=0}^{l_1} m_s^1\overline{x_s^1}+\ldots+\mu_k\sum_{s=1}^{l_k} m_j^k\overline{x_s^k}=\sum_{i=0}^{k}\sum_{s=0}^{l_i}\mu_i m_s^i\overline{x_s^i}$$; We

put $a_s^i=m_s^i\mu_1\ge 0$, $\sum_{i=0}^{k}\sum_{s=0}^{l_i} a_s^i=1$. Hence, $z=\sum_{i=0}^{k}\sum_{s=0}^{l_i} a_s^i\overline{x_s^i}$, $\overline{x_s^i}\in X$. Hence, $z\in X(X)$ that is the mapping $\psi\in Z(Z(X))\to Z(X)$ is defined. The mapping Ψ_x is continuous because Ψ_x is linear.

A simple check shows that for the triple $Z=Z,\Psi,\eta_X$ satisfies the following conditions:

(a) $\Psi_X\circ Z(\eta_X)=id_{Z(X)}$;

(b) $\Psi_X\circ\eta_{Z(x)}=id_{Z(X)}$;

(c) $\Psi_X\circ Z(\Psi_X)=\Psi_X\Psi_{Z(X)}$;

Z: $S\to S$ functor, η_Z: $id\to Z$ (one) and Ψ: $Z^2\to Z$ (multiplication) natural transformation. Hence, the triple $T=\langle Z,\Psi,\eta_X\rangle$ is monadic. Theorem 4 is proved.

Theorem 5. *For any stratifiable space X, the space Z(X) is convex and the subspace $Z_\nabla(X)$ is homotopically dense in Z(X).*

Proof. Let $X\in S$. Two cases are possible : a) X is finite. b) X is infinite. In case a) the space $Z(X)$ consists of one simplex $\sigma=<x_0, x_1,..., x_n>$. This simplex σ is convex and the subspace $Z_\nabla(X)=Z(X)\setminus X=\sigma\setminus\{x_0, x_1,..., x_n\}$ is homotopy and dense in $Z(X)$. Let X be infinite. In this case, the space $Z(X)$ is convex. Indeed, for any points $z_0\in Z(X)$, $z_1\in Z(X)$, the segment $[z_0,z_1]$ lies entirely in the space $Z(X)$, i.e. $[z_0,z_1]\subset Z(X)$. Now we can that the subspace $Z_\nabla(X)$ is homotopically dense in $Z(X)$.

We construct the desired homotopy $h(z,t)$:$Z(X)\times[0,1]\to Z(X)$ supposing $h(z,t)=(1\text{-}t)\ z+t\cdot z_0$, where $z\in Z(X)$, $t\in[0,1]$, $z_0=m_0\overline{x}_0+...+m_n\overline{x}_n,\sum_{i=1}^{n}m_i=1$, $m_i>0$, $m_i\neq 0$.

If $t=0$, then $h(z,0)=(1\text{-}0)\ z+0\cdot z_0=z$, i.e. $h(z,0)=id_{z(x)}$.

If $t\in(0,1]$, that is, $t>0$, $t\le 1$, then the point $h(z,1)=(1\text{-}t)\ z+t\cdot\overline{z}_0\in Z_1(X)$, i.e. $h(z,0)=id_{z(x)}$, i.e. point $h(z,1)\in Z_\nabla(X)$. This means that the subspace $Z_\nabla(X)$ is homotopically dense in $Z(X)$. The theorem is proved.

A subset $A\subset X$ of the space X is said to be homotopically dense in X [9] if there exists a homotopy h:$X\times[0,1]\to X$ such that $h(x,0)=id_x$ and $h(X\times(0,1])\subset A$. Therefore, the subset $A\subset X$ is called homotopy negligible [9] if the complements $X\setminus A$ is homotopy dense in X. The above theorem implies.

Corollary. *For any stratifiable space X, the space $Z_1(X)$ itself is homotopically negligible in Z(X)*

Theorem 6. *Let X be a finite-dimensional St-space. Then $Z_n(X)$ is a finite-dimensional St-space. Moreover, the inequality $\dim Z_n(X)\le\dim X^n+n\text{-}1\le n\dim X+n\text{-}1$ is true.*

Proof. Let X be a *St*- space and $\dim X<\infty$. Note that for any $n\in N$ the space $Z_n(X)$ is also *St*-a space, therefore, $Z_n(X)$ is a perfectly normal paracompact space. It follows from the results of [8,10] that

$$dim\ X^n \times \sigma^{n-1} \leq dimX^n + dim\ \sigma^{n-1} \leq n\ dim\ X + n\text{-}1$$

Through induction on n, =

$$dim\ Z_n(X) \leq n\ dim\ X + n\text{-}1 \quad (3)$$

For $n = 1$, this inequality (3) is valid, since the space $Z_1(X)$ is homeomorphic to X by definition. those $Z_1(X)$; X.

Let the inequality $dim\ Z_{n-1}(X) \leq (n\text{-}1)\ dim\ X + dim\ Z_{n-1}\,(\overline{n\text{-}1})$. The theorem is proved.

First, it is demonstrated that $rd_{Zn}(X)\backslash Z_{n-1}(X)) \leq n\ dim\ X + n\ \text{-}1$, where $rd_X A$ is the relative dimension of the subspace A in the space [7]. The subspace $Z_n(X)\backslash Z_{n-1}(X)$ is also St- space, hence it is paracompact. The space $Z_n(X)$ at the points of the set $Z_n(X)\backslash Z_{n-1}(X)$ is locally homeomorphic to the open subspace of the product $X^n \times \sigma^{n-1}$. From Theorem 21 of [7] Dowker-Nagami we obtain that $rd_{Zn(X)}(Z_n(X))\backslash Z_{n-1}(X)) \leq n\ dim\ X + n\text{-}1$.

Then, applying another Dowker's theorem 10 [7], we obtain that

$$dim\ Z_n(X) \leq n \cdot dim\ X + n\text{-}1.$$

The theorem 6 is proved.

A space X is weakly star countable if for each open cover U of X there exists a countable subset F of X such that $(\bigcup_{x\in F} St(x,U)=X$.

Theorem 7. *If X is a weakly countable St-space, then* $Z_n(X)$ *is also weakly countable.*

Theorem 8. *For any weakly countable or finite-dimensional St-space, the space Z(X) is weakly countable.*

Theorem 9. *For any infinite St- space X, the space Z(X) is infinite-dimensional.*

Definition. For the subset of the normal space X, we write [7] $rd_X A \leq n$ (relative dimension A in $X \leq n$), if for any set F lying in A and closed in X the inequality $dim\ F \leq n$, $n = -1, 0, 1, 2, \ldots$.

Theorem 10 [7]. *If in the normal space X there exists a closed set F such that* $dim\ F \leq n$ *and* $rd_X(X\backslash F) \leq n$, *then* $dim\ X \leq n$.

Theorem 11 [7]. *For paracompact the following is always true*

$$\ell oc\ dim\ X = dim\ X$$

Definition [7]. For a normal space X, we assume $\ell oc\ dim\ X \leq n$ if for any point $x \in X$ there exists a neighborhood Ox for which $dim[Ox] \leq n$, $n = 0, 1, 2, \ldots$. (For empty X we get $\ell oc\ dim\ X = -1$).

CONFLICTS OF INTEREST

The authors declare that they have no conflict of interest.

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