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2-LOCAL AUTOMORPHISMS AND ANTI-AUTOMORPHISMS ON REAL AND COMPLEX AW*-ALGEBRAS

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ABSTRACT

This study focuses on 2-local automorphisms and anti-automorphisms on real and complex matrix algebras over unital real and complex Banach algebras. It is found that any 2-local anti-automorphisms on AW*-algebra without finite type I direct summands is an anti-automorphism. It is also found that any 2-local automorphisms on real AW*-algebra without finite type I direct summands is an automorphism.

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INTRODUCTION

Let N be an *-algebra. A linear operator α: $N\to N$ is called a **-automorphism* (respectively, a *-antiautomorphism), if $\alpha(a^*) = \alpha(a)^*$ and $\alpha(ab) = \alpha(a)\alpha(b)$ (respectively, $\alpha(ab) = \alpha(b)\alpha(a)$), for all $a,b\in N$. Each unitary (or invertible) element $u\in N$ implements a *-automorphism Adu on N defined as $Adu(a):=uau^{-1}$, $a\in N$. Such *-automorphisms are said to be *inner *-automorphisms*. A linear map Θ: A→A is called a *local *-automorphism* (respectively, a *local *-antiautomorphism*), if for every $a\in N$, there is a *-automorphism (respectively, a *-antiautomorphism) α_a of N such that $\Theta(a) = \alpha_a(a)$. Now, a map Θ: A→A (not linear in general) is called a *2-local *-automorphism* (respectively, *2-local *-antiautomorphism*), if for every $a,b\in N$, there exists a *-automorphism (respectively, a *-antiautomorphism) $\alpha_{a,b}$: $N\to N$ such that $\Theta:(a) = \alpha_{a,b}(a)$ and $\Theta:(b) = \alpha_{a,b}(b)$. If in the definition of a 2-local *-automorphism (respectively, a *-antiautomorphism) we require linearity of mapping Θ, then, it is easy to demonstrate that it becomes a *-automorphism (respectively, a *-antiautomorphism). Moreover, it is also obvious that, for $n \geq 3$, n-local *-automorphisms (respectively, *-antiautomorphisms) are *-automorphisms (respectively, *-antiautomorphisms).

In [1] R.V. Kadison studied the concept of local derivations and demonstrated that local derivations of von Neumann algebras (i.e. W*-algebras) are derivations. Building on this, B.E. Johnson, in his study [2], generalised this result for C*-algebras. D.Larson and A.Sourour in [3] introduced and considered the notion of local automorphisms, proving that local automorphisms on $B(X)$ are automorphisms, where X represents an infinite-dimensional complex Banach space and $B(X)$ denotes the algebra of all bounded linear operators on X. For real C*- and W*-algebras, the works of U. Karimov, such as [4], considered local derivations and local *-automorphisms. In [5] P. Semrl investigated 2-local automorphisms, specifically describing their properties on $B(H)$, where H is the infinite-dimensional separable complex Hilbert space and $B(H)$ is the algebra of all bounded linear operators on H. Later, in [6], the same description was extended to the finite-dimensional case. A real analogue of these results was partially obtained in [7], where H is a real Hilbert space.

This paper focuses on 2-local automorphisms and anti-automorphisms on real and complex matrix algebras over unital real and complex Banach algebras. The results demonstrated that any 2-local anti-automorphism on a (complex) AW*-algebra without finite type I direct summands is an anti-automorphism. Additionally, this paper establishes that any 2-local automorphism on real AW*-algebra without finite type I direct summands is an automorphism.

It should be noted here that the study of 2-local anti-automorphisms, generally speaking, cannot be reduced to the study of 2-local automorphisms. Indeed, if $(e_\alpha)_{\alpha\in I}$ is some orthonormal basis for H, then, map j: $\sum\lambda_\alpha e_\alpha \to \sum\bar{\lambda}_\alpha e_\alpha$ is the conjugation of H, and hence, we can construct a sufficient many (continuum) of pairwise non-conjugate involutive anti-automorphisms $J(.)^* J$ of C*- and AW*-algebras. Therefore, it suffices for us to consider a 2-local anti-automorphism Θ such that for any x and y there exists conjugation $J = J_{x,y}$ with $J^2 = 1$ (or $J^2 = -1$) and $\Theta(x) = Jx^*J$, $\Theta(y) = Jy^*J$. It is clear that the study of Θ cannot be reduced to the study of 2-local automorphisms.

Next, it is essential to note that there are examples of non-trivial 2-local automorphisms and anti-automorphisms. Let A be a subalgebra of $M_3(\mathbb{C})$, generated by elements (matrices) I, E_{12}, E_{13}, where I is a unit, and E_{12}, E_{13} are the matrix units of algebra $M_3(\mathbb{C})$. It is directly shown that a general form of automorphism of algebra A is the linear extension of θ acting on these elements as follows:

$$\theta(I) = I, \quad \theta(E_{12}) = aE_{12} + bE_{13}, \quad \theta b E_{13} = cE_{12} + dE_{13}, \text{ where } ad - bc \neq 0.$$

The mapping Θ: $A \to A$ is defined as:

$$\Theta(a_{11}I + a_{12}E_{12} + a_{13}E_{13}) := a_{11}I + a^3_{12}E_{12} + a^3_{13}E_{13}.$$

It's obvious that Θ is non-linear, thus, not an automorphism. But it is a 2-local automorphism. Let $A = a_{11}I + a_{12}E_{12} + a_{13}E_{13}$ and $B = b_{11}I + b_{12}E_{12} + b_{13}E_{13}$. If A and B are linearly dependent, then, it is not difficult to find the desired automorphism. If A and B are linearly independent, then, their images $\Theta(A)$ and $\Theta(B)$ are also linearly independent. Moreover, the elements:

$$\left\{\begin{pmatrix} a_{12} \\ a_{13} \end{pmatrix}, \begin{pmatrix} b_{12} \\ b_{13} \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} a^3_{12} \\ a^3_{13} \end{pmatrix}, \begin{pmatrix} b^3_{12} \\ b^3_{13} \end{pmatrix}\right\}$$

are bases for $\mathbb{C}^2$. Then there is a non-singular change of the basis matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{13} \end{pmatrix} = \begin{pmatrix} a^3_{12} \\ a^3_{13} \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b_{12} \\ b_{13} \end{pmatrix} = \begin{pmatrix} b^3_{12} \\ b^3_{13} \end{pmatrix}.$$

Thus, there is an automorphism $\beta_{A,B}$ of A, defined by $a, b, c, d,$ such that $\beta_{A,B}(A) = \Theta(A)$ and $\beta_{A,B}(B) = \Theta(B)$.

PRELIMINARIES

Let H be a complex Hilbert space. Let $B(H)$ be the algebra of all bounded linear operators on H. The weak (operator) topology on $B(H)$ is the locally convex topology, generated by the seminorm of the form: $\rho(a) = |(\xi, a\eta)|$, $\xi, \eta \in H$ $a \in B(H)$. A weakly closed *-subalgebra M containing identity operator 1 in $B(H)$ is called a *W*-algebra*. Recall that W^* -algebras are also called von Neumann algebras. The centre $Z(M)$ of a W^*-algebra M is the set of elements of M, commuting with each element from M. Elements of $Z(M)$ are called central elements. A W^*-algebra M is called a factor, if the centre consists of the complex multiples of 1, i.e. $Z(M) = \{\lambda 1, \lambda \in \mathbb{C}\}$.

Let A be a Banach *-algebra over field C. Algebra A is called a C*-*algebra*, if $\|xx^*\|=\|x\|^2$, for any $x \in A$. A real Banach *-algebra R is called a *real* C*-*algebra*, if $\|xx^*\|=\|x\|^2$ and an element $1+xx^*$ is invertible for any $x \in R$. It is easy to see that R is a real C*-algebra if and only if a norm on R can be extended onto the complexification A = R +iR of algebra R so that algebra A is a C*-algebra (see [8, 5.1.1]). A real *-subalgebra is called a real *W*- algebra* if it is closed in the weak operator topology, $1 \in R$ and $R \cap iR = \{0\}$. The smallest complex W*-algebra M containing R coincides with its complexification, $R+i$R i.e. $M = R+iR$. We say that a real W*-algebra R is of the type I_{fin}, I_∞, II_1, II_∞, or III_λ $(0 \le \lambda \le 1)$ if the enveloping W*-algebra $R+iR$ has the corresponding type in the ordinary classification of W*-algebras.

Now, recall that some definitions and facts from the theory of complex and real AW*-algebras (see [9]). Let A be a real or complex *-algebra. Let S be a nonempty subset of A. We consider the sets

$$R(S) = \{x \in A \mid sx = 0 \text{ for all } s \in S\},\ L(S) = \{x \in A \mid xs = 0 \text{ for all } s \in S\}$$

Set $R(S)$ is called the right-annihilator of S and set $L(S)$ is called the left-annihilator of S. A *-algebra A is called a Baer *-algebra if for $\forall S \subset A$, $S \neq \varnothing$, $R(S) = gA$ for an appropriate projection g. Since $L(S) = (R(S^*))^* = (hA)^* = Ah$, the definition is symmetric and can be given in terms of the left-annihilator and a suitable projection h, here $S^* = \{s^* | s \in S\}$. A complex (respectively, real) C*-algebra A, which is a Baer *-algebra is called an *AW*-algebra* (respectively, real *AW*-algebra*). It is easy to show that any W*-algebra is an AW*-algebra (see [9], Proposition 9, page 24). In [10], the author showed that the converse is not true. As is known that if R is a real C*- or W*-algebra, then, its complexification R+iR is C*- or W*-algebra, respectively. However, this is not true for real AW*-algebras, i.e., there exists a real AW*-algebra for which the complexification is not an AW*-algebra [11, Proposition 4.2.3]. The theory of C*-, AW*- and W*-algebras can also be found in detail in the works [8], [9], [11], [12], [13], [14], [15], and [16].

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Recall that if A is an algebra and θ: A→A is a linear map with the property $\theta(x\text{o}y) = \theta(x)\text{o}\ \theta(y)$, then, θ is called a Jordan *-automorphism, where $x\text{o}y = (xy + yx)/2$ is the Jordan multiplication. It is obvious that if $\theta(x^2) = \theta(x^2)$ (for all x), then, by the linearity of θ, we obtain $\theta(xy+yx) = \theta(x)\ \theta(y) + \theta(y)\ \theta(x)$ and hence, θ is a Jordan *-automorphism. It is easy to show that if Θ is 2-local *-automorphism (respectively, *-anti-automorphism) of A, then, $\Theta(x^2) = \Theta(x^2)$, $\forall x \in$A. Therefore, if 2-local *-automorphism (respectively, *-anti-automorphism) θ is linear, then, θ is a Jordan *-automorphism.

The main result of this article is the following theorem.

Theorem 3.1. If A is a complex (respectively real) AW*-algebra without finite type I direct summands, then every 2-local anti-automorphism of A is an anti-automorphism. In the real AW*-algebra case, it is assumed that its complexification is an AW*-algebra, every 2-local automorphism of is an automorphism.

In order to prove the theorem, the representation of the real AW*-algebra in the form of matrix algebras over a unital real Banach algebra is used, which has the following properties.

(J): if α is a Jordan automorphism of A, then, there is a decomposition $A = A_1 \oplus A_2$ of A with 1) the map $A\breve{A}x\to p_1(\alpha(x))\in A_1$ being a homomorphism; and, 2) the map $A\breve{A}x\to p_2(\alpha(x))\in A_2$ being an anti-homomorphism. Here, p_1 is a projection from A onto A_i (i = 1,2).

(M): $\exists x,y \in A$ with $xy = 0$ and $yx \neq 0$.

As shown in [17], for any unital complex Banach algebra A with properties (J) and (M), every 2-local automorphism of $M_{2^n}(A)$ is an automorphism, where $M_{2^n}(A)$ is the algebra of 2^n-th order matrix over algebra A.

The proof of Theorem 2.1 from [17б] can be done in a similar way for 2-local anti-automorphisms of $M_{2^n}(A)$, and moreover, this proof without changes passes for real Banach algebras as well. Therefore, we can formulate this result for real Banach algebras.

Theorem 3.2. If A is a unital complex (respectively, real) Banach algebra with the properties (J) and (M), then any 2-local anti-automorphism (respectively, 2-local automorphism) on $M_{2^n}(A)$ is an anti-automorphism (respectively, automorphism).

In order to describe 2-local anti-automorphisms and automorphisms on complex and real AW*-algebras, we apply Theorem 3.2.

Proof of Theorem 3.1. Firstly, by Theorem 3.3 from [18] (see also Theorem 3.2.3 [15]), every C*-algebra, and hence, AW*-algebra, has the property (J). Then, any real C*-algebra, and hence, real AW*-algebra, also has the property (J). Indeed, let A be a real AW*-algebra and let θ be a Jordan automorphism on A. We extend it by linearity to the enveloping C*-algebra $A_C = A + iA$ as $\Theta\ (x + iy) = \theta(x)+ i\theta(y)$. It can be shown directly that Θ is also Jordan automorphism of A_C. Then, the algebra A_C has a decomposition $A_C = A^1{}_C \oplus A^2{}_C$, such that the map $A_C\breve{A}x\to p_1(\Theta(x))\in A^1{}_C$ is a homomorphism, and the map $A_C\breve{A}x\to p_2(\Theta(x))\in A^2{}_C$ is an anti-homomorphism. Here, p_i: $A_c\to A^i{}_C$ is a projection (i=1,2). Then, for $A_i = A^i_C\cap A$ (i = 1,2) the algebra A also has a decomposition $A = A_1 \oplus A_2$ such that the map $A\breve{A}x\to q_1(\theta(x))\in A_1$ is a homomorphism, and the map $A\breve{A}x\to q_2(\theta(x))\in A_2$ is an anti-homomorphism, here, $q_i = p_i|_{A_i}$ is a projection: $A \to A_i$(i = 1,2).

By Proposition 4.4.3 [11,] any non-abelian real AW*-algebra can be decomposed along a central projection into the direct sum of real AW*-algebras of types I_n, $n \geq 2$, I_∞,II and III. According to the conditions of the theorem, A is an algebra without finite I-type direct summands. Hence, there are mutually orthogonal central projections z_1, z_2, $z_3\in A$ with $A = z_1A\oplus z_2A\oplus z_3A$. Here, algebras $A = z_1A$, z_2A and z_3A have types I_∞,II and III, respectively.

Similarly, following the scheme of the proof of Lemmas 4.5; 4.12 [1] and using the methods developed in [11, §§4.3-4.7], and also applying the result to each term, it is implied that element $z_i \in z_iA$(i = 1,2,3) can be represented as a sum of orthogonal, mutually equivalent projections $e_1^{(i)}$, $e_2^{(i)}$, $e_3^{(i)}$, $e_4^{(i)} \in z_iA$(i=1,2,3). Put $e_k = \sum_{i=1}^{3} e_k^{(i)}$, $k=\overline{1,4}$. Hence, the mapping $x\to\sum_{i,j=1}^{4} e_ixe_j$ defines an isomorphism: $A\to M_4(B)$,where $B = e_{1,1}Ae_{1,1}$. It is not difficult to show that algebras $B + iB$ and B are complex and real C*-algebras, respectively. According to [9, Proposition 8 (iii), 23p.], for any projection $e\in A_c$, algebra eA_ce is an AW*-algebra, as $A_c = A+iA$ is an AW*-algebra. Then, algebra B=$e_{1,1}Ae_{1,1}$is also a real AW*-algebra (see Proposition 4.3.1[11]). Moreover, its complexification $B+iB = e_{1,1}A_c e_{1,1}$ is also an AW*-algebra. Hence, B has the property (J). Moreover, B also satisfies the condition (M), as it contains a subalgebra isomorphic to $M_2(\mathbb{R})$. For instance, for matrices:

$$x = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

We have $yx \neq 0$, $xy \neq 0$. Then, according to Theorem 3.2, every 2-local automorphism of A is an automorphism. Similarly, in complex cases, algebra B also has the properties (J) and (M). Hence, again, according to Theorem 3.2, every 2-local anti-automorphism of A is an anti-automorphism. Theorem proved.

Remark 3.1. According to the condition of the theorem, we require that the complexification $A+iA$ is an AW*-algebra. In addition, the definition of a real C*-algebra requires the condition that an element $1+xx^*$ is invertible (for all x). But in [9] (see Exercise 14A), this condition is absent, i.e., the reversibility of element $1+xx^*$ is not required.

Therefore, naturally, the following problem arises.

Problem: Suppose that A is a real Baer *-ring. Let:

(a) A be a real Banach *-algebra such that $\|xx^*\|=\|x^2\|$, $(x\in A)$, or,

(b) A be a real AW*-algebra for which its complexification is not necessarily an AW*-algebra.

Then, is every 2-local automorphism (anti-automorphism) of A an automorphism (respectively, an anti-automorphism)?

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

REFERENCES

[1] Kadison, R. V. (1990). Local derivations. *Journal of Algebra*, *130*(2), 494-509. https://doi.org/10.1016/0021-8693(90)90095-6

[2] Johnson, B. E. (2000). Local derivations on C*-algebras are derivations. *Transactions of the American Mathematical Society*, *353*(1), (2000), 313-325. https://www.ams.org/journals/tran/2001-353-01/S0002-9947-00-02688-X/S0002-9947-00-02688-X.pdf

[3] Larson, D. R., & Sourour, A. R. (1990). Local derivations and local automorphisms. *Proceedings of Symposia in Pure Mathematics*, *51,* 187-194. DOI: https://doi.org/10.1090/pspum/051.2

[4] Karimov, U. (2020). On the local derivations on real W*-algebras. *Bulletin of the Institute of Mathematics*, *4*, 23-28.

[5] Semrl, P. (1997). Local automorphisms and derivations on $B(H)$. *Proceedings of the American Mathematical Society*, *125*(9), 2677-2680.

[6] Kim, S. O. & Kim, J. S. (2004). Local automorphisms and derivations on M_n. *Proceedings of the American Mathematical Society*, *132*, 1389-1392.

[7] Rakhimov, A. A., Nazarov, Kh. A., & Khoshimov, A. S. (2020). 2-Local *-automorphisms on real W*-algebra $B(H_r)$. *Uzbek Mathematical Journal*, *4,* 111-116. DOI: 10.29229/uzmj.2020-4-12

[8] Li, B.-R. (2003). *Real operator algebras* (pp. 241). World Scientific Publishing Co. Pte. Ltd.

[9] Berberian, S.K. (1972). *Baer *-rings* (Vol. 195). Berlin Heidelberg: Springer-Verlag.

[10] Dixmier, P. J. (1951). Sur certains espaces consid.er.es par M. H. Stone (*). *Summa Brasiliensis Mathematicae*, *2,* 151-182.

[11] Ayupov, Sh. A., & Rakhimov, A. A. (2010). *Real W*-algebras, Actions of groups and Index theory for real factors* (pp. 138). Beau-Bassin, Mauritius: VDM Publishing House Ltd.

[12] Ayupov, Sh. A., Rakhimov, A. A., & Usmanov, Sh. M. (1997). *Jordan, real and lie structures in operator algebras* (pp. 235). Springer Dordrecht.

[13] Dadakhodjaev, R. A., & Rakhimov, A. A. (2021). 2-Local derivations of real AW*-algebras are derivation. *Positivity*, *25*, 13511356.

[14] Rakhimov, A. A., & Nurillaev, M. E. (2018). On property of injectivity for real W*-algebras and JW-algebras. *Positivity*, *22,* 1345-1354.

[15] Brattelli, O., & Robinson, D. (2002). *Operator algebras and quantum statistical mechanics* (2nd ed., pp. 517). Springer-Verlag Berlin Heidelberg New York.

[16] Takesaki, M. (1979). *Theory of operator algebras I*. Springer-Verlag, New York.

[17] Ayupov, Sh. A., & Kudaybergenov, K. K., & Kalandarov, T. (2019). 2-local automorphisms on AW*-algebras. In *Positivity and Noncommutative Analysis*. Birkhäuser Cham.

[18] Stormer, E. (1965). On the Jordan structure of C*-algebras. *Transactions of the American Mathematical Society*, *120*, 438-447.