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# **MATHEMATICAL MODELLING OF HEAT TRANSFER IN BAKING COOKIES**

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## **INTRODUCTION**

Bakery products are very popular among the people and the bakery industry is witnessing significant growth, with the market expected to increase from US\$ 416.36 billion in 2021 to US\$ 590.52 billion by 2028. In Malaysia, the sales of bakery products reached approximately RM12.38 billion in 2021 [1]. The process of baking begins with a fluid batter, which undergoes the baking process to transform into a porous solid product [2]. Heat transfer is a process that involves the exchange of internal energy between two substances [3]. The First Law of Thermodynamics explains heat transfer, stating that energy can only be transferred, but cannot be created or destroyed [4]. In recent years, numerous publications have focused on describing the baking process of various bakery products, including cakes, cookies, and bread [5]. Due to the complexity of the baking process and the involvement of multiple factors, researchers have explored various aspects to understand the relationships between temperature, moisture content, volume change, weight loss, and colour of baked goods [6].

Heat transfer is an essential part of daily life, often unnoticed in our routines. Baking, a familiar heat transfer phenomenon, may not be fully comprehended by everyone. During the COVID-19 pandemic, with people staying at home, many turned to new hobbies, and baking or cooking became the second most popular activity among Americans, comprising 36% of their leisure pursuits [7]. However, incorrect steps or calculations can lead to baking failures. Therefore, it is essential to understand the steps involved in the baking process. When the dough is put into the oven, the food surface's temperature increases and heat energy diffuses into the food through conduction. This process leads to the evaporation of water from the dough surface, resulting in a low moisture content and the formation of a golden-brown crust [8]. Mathematical models offer a detailed understanding of processes that might not be evident from simple data charts. Therefore, a mathematical model for mass and heat transfer in the baking of cookies will be examined.

This study aims to discretise the Partial Differential Equation (PDE) using the Finite Difference Method (FDM) for the heat transfer phenomena in cookies during the baking process. This study also aims to determine the temperature and moisture transfer properties using the model from objective 1. This study focused on modelling heat and mass transfer in cookies during the cookie-baking process, assuming that the heat and mass transfer process only occurs inside the cookie. The baking stage, which involves convection and conduction as modes of heat transport, and convection and diffusion for mass transport, will be covered in this research. The main focus will be on analyzing the change in temperature and moisture content of the cookie during baking. The relationship between temperature and moisture content with the thickness of the cookie, the radial distance from the side pan, and the time taken to bake the cookie will be explored through numerical solutions based on the mathematical model, while [9] solved the problem experimentally.

#### **METHODOLOGY**

In this study, both moisture and temperature transfer of the cookie will be considered in both the radial and axial directions. For the radial direction, both the right and left sides of the cookie next to the pan are represented as  $r = 0$ . This study aims to discuss the diffusion of water and heat transfer between the central and outer regions of the cookie. The cookie will be treated as a cylinder and divided into two parts, the central  $(C<sub>s</sub>)$  and outer regions  $(O<sub>s</sub>)$  as illustrated in the figure below.



Figure 1: An illustration of the cookie

According to Figure 1, the radial distance from the side of the pan ranges from 0 to  $r_f$ . The radial distance between 0 to  $r_o$  is the outer section, while  $r_o$  to  $r_f$  is the central section. As the baking process starts, the temperatures at the left and bottom of the cookie are at the boiling temperature. The modelling work assumes a uniform heat transfer coefficient and moisture diffusivity across the entire cookie.

#### **Governing Equations**

The equations that govern the heat and mass transfer inside the butter cookie, as in Equations (1) and (5), are given in terms of spherical coordinates as follows (refer to [9]):

$$
\rho (CT + \lambda) \frac{\partial X}{\partial t} + \rho C \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial r^2} - \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right)
$$
(1)

$$
T(r, z, 0) = 20^{\circ}C \tag{2}
$$

$$
\mathbf{T}(r,0,t) = 130^{\circ}\mathbf{C} \tag{3}
$$

$$
\Gamma(0, z, t) = 130^{\circ} \text{C}
$$
 (4)

where  $\rho$  is the density of the product [kg/m<sup>3</sup>], C is the specific heat capacity [J/(kgK], T is the temperature  $[^{\circ}C]$ ,  $T_0$  is the initial temperature  $[^{\circ}C]$ ,  $r$  is the radial direction  $[cm]$  and  $z$  is the thickness [cm]. The heating of the product will follow Fourier's Law and the heat transfer equation is shown in Equation (1), the Initial Condition (IC) in Equation (2), and the Boundary Conditions (BC) are shown in Equations (3) and (4).

The water transport will follow Fick's Second Law and the mass transfer equation is shown as (refer to [9]):

$$
\frac{\partial X}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{D}{1 - X} \frac{\partial X}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{D}{1 - X} \frac{\partial X}{\partial z} \right) \tag{5}
$$

$$
X(r, z, 0) = 0.6 \tag{0}
$$

$$
X(r, 0, t) = 0.35\tag{7}
$$

$$
X(0, z, t) = 0.2\tag{8}
$$

where *D* is the diffusivity of water in the cookie  $[m^2/s]$ , *r* is the radial direction [cm], *z* is the thickness  $[\text{cm}]$ , and *X* is the moisture content  $[\text{kg water/kg solids}]$ . Based on Equation (5), the Initial Condition (IC) in Equation (6) and Boundary Conditions (BC) are shown in Equations (7) and (8).

#### **Finite Difference Method**

The Finite Difference Method (FDM) is used to discretise Partial Differential Equations (PDE) and then solve two-dimensional heat and mass Equations (1) and (5) in the cookie. FDM was chosen for this study because it is relatively straightforward to implement a problem with simple geometry, such as a cylinder, which is the geometry considered in this study. This method is easier compared with the Finite Element Method (FEM) or Finite Volume Method (FVM). FDM is also more efficient in terms of computation compared with FEM or FVM. Both heat and mass transfer equations involve variables *r, z* and *t.* The continuous and unbounded *r, z* and *t* are converted into a bounded and discrete set of grid points. Next, the partial derivatives are approximated, which transforms the PDE into a system of Ordinary Differential Equations (ODE). In this study, the Forward Time-Centred Space (FTCS) method will be used to solve the PDE. The FTCS method is based on the central difference in both two spaces and the forward Euler method in time. The approximation of the PDE according to Equations (1) and (5) is as follows:

 $(6)$ 

$$
\frac{\partial \mathbf{T}}{\partial t} = \frac{\mathbf{T}_{i,j}^{n+1} - \mathbf{T}_{i,j}^n}{\Delta t}
$$
\n(9)

$$
\frac{\partial X}{\partial t} = \frac{X_{i,j}^{n+1} - X_{i,j}^n}{\Delta t}
$$
\n(10)

$$
\frac{\partial \mathbf{T}}{\partial r} = \frac{\mathbf{T}_{i+1,j}^n - \mathbf{T}_{i-1,j}^n}{2\Delta r}
$$
(11)

$$
\frac{\partial X}{\partial r} = \frac{X_{i+1,j}^n - X_{i-1,j}^n}{2\Delta r}
$$
\n(12)

$$
\frac{\partial^2 T}{\partial r^2} = \frac{T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n}{\Delta r^2}
$$
(13)

$$
\frac{\partial^2 X}{\partial r^2} = \frac{X_{i+1,j}^n - 2X_{i,j}^n + X_{i-1,j}^n}{\Delta r^2}
$$
 (14)

$$
\frac{\partial X}{\partial z} = \frac{X_{i,j+1}^n - X_{i,j-1}^n}{2\Delta z} \tag{15}
$$

$$
\frac{\partial^2 T}{\partial z^2} = \frac{T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n}{\Delta z^2}
$$
(16)

$$
\frac{\partial^2 X}{\partial z^2} = \frac{X_{i,j+1}^n - 2X_{i,j}^n + X_{i,j-1}^n}{\Delta z^2}
$$
 (17)

#### **Heat Transfer Equation**

To solve the heat transfer equation, we substitute Equations (9), (10), (11), (13) and (15) into Equation (1). Then, we have:

$$
\rho C \frac{T_{i,j}^{n+1} - T_{i,j}^n}{\Delta t} = k \left( \frac{T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n}{\Delta r^2} - \frac{1}{r} \frac{T_{i+1,j}^n - T_{i-1,j}^n}{2\Delta r} + \frac{T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n}{\Delta z^2} \right) - \rho (C\Gamma + \lambda) \frac{X_{i,j}^{n+1} - X_{i,j}^n}{\Delta t}
$$
(18)

Simplifying the equation above, we have:

$$
T_{i,j}^{n+1} = T_{i,j}^n + \frac{\Delta t}{\rho C} \left[ k \left( \frac{T_{i+1,j}^n - 2T_{i,j}^n + T_{i-1,j}^n}{\Delta r^2} - \frac{1}{r} \frac{T_{i+1,j}^n - T_{i-1,j}^n}{2\Delta r} + \frac{T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n}{\Delta z^2} \right) - \rho (C T + \lambda) \frac{X_{i,j}^{n+1} - X_{i,j}^n}{\Delta t} \right]
$$
\n(19)

Then, thermal conductivity, k, in the radial and axial directions are different to reduce the error [3]. Thermal conductivity in the radial direction is denoted as  $k_R$ , while  $k_z$  denotes the axial direction. Equation (19) is rearranged by multiplying the density in the second term with the initial moisture and oil content  $(0.5 + X_{i,j}^n + Y)$  as this term is influenced by the temperature. Therefore, the heat transfer Equation (19) becomes (refer to [9]):

$$
T_{i,j}^{n+1} = T_{i,j}^{n} + \frac{\Delta t}{\rho(0.5 + X_{i,j}^{n} + Y)C} \left[ \left( k_R \frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta r^2} - \frac{k_R T_{i+1,j}^{n} - T_{i-1,j}^{n}}{2\Delta r} + k_Z \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta z^2} \right) - \rho (CT + \lambda) \frac{X_{i,j}^{n+1} - X_{i,j}^{n}}{\Delta t} \right].
$$
\n(20)

where *Y* is the oil content of the cookies.

### **Mass Transfer Equation**

For the mass transfer equation, the differentiation for both terms on the Right-Hand Side (RHS) of Equation (5) needs to be solved using the product rule as stated as follows:

$$
\frac{\partial}{\partial r}(uvw) = u'vm + uv'm + uvm
$$

For the first term, after differentiating and substituting the solution into Equation (5), we have:

$$
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{D}{1-X}\frac{\partial X}{\partial r}\right) = \frac{D}{r(1-X)}\frac{\partial X}{\partial r} + \frac{D}{(1-X)^2}\left(\frac{\partial X}{\partial r}\right)^2 + \frac{D}{1-X}\frac{\partial^2 X}{\partial r^2}
$$
(21)

For the second term, we differentiate and obtain the solution as follows:

$$
\frac{\partial}{\partial z} \left( \frac{D}{1 - x} \frac{\partial X}{\partial z} \right) = \frac{D}{1 - x} \frac{\partial^2 X}{\partial z^2} + \frac{D}{(1 - x)^2} \left( \frac{\partial X}{\partial z} \right)^2 \tag{22}
$$

Then, we substitute Equations (21) and (22) into Equation (5). Hence, the mass transfer equation can be simplified as follows:

$$
\frac{\partial X}{\partial t} = \frac{D}{r(1-X)} \frac{\partial X}{\partial r} + \frac{D}{(1-X)^2} \left(\frac{\partial X}{\partial r}\right)^2 + \frac{D}{1-X} \frac{\partial^2 X}{\partial r^2} + \frac{D}{1-X} \frac{\partial^2 X}{\partial z^2} + \frac{D}{(1-X)^2} \left(\frac{\partial X}{\partial z}\right)^2
$$
\n(23)

Next, we substitute Equation (10), (12), (14), (15), and (17) into Equation (23). Therefore, we have:

$$
X_{i,j}^{n+1} = X_{i,j}^{n} + \Delta t D \left( \frac{1}{r(1 - X_{i,j}^{n})} \frac{X_{i+1,j}^{n} - X_{i-1,j}^{n}}{2\Delta r} + \frac{1}{(1 - X_{i,j}^{n})^{2}} \left( \frac{X_{i+1,j}^{n} - X_{i-1,j}^{n}}{2\Delta r} \right)^{2} + \frac{1}{(1 - X_{i,j}^{n})} \frac{X_{i+1,j}^{n} - 2X_{i,j}^{n} + X_{i-1,j}^{n}}{\Delta r^{2}} + \frac{1}{(1 - X_{i,j}^{n})} \frac{X_{i,j+1}^{n} - 2X_{i,j}^{n} + X_{i,j-1}^{n}}{\Delta z^{2}} + \frac{1}{(1 - X_{i,j}^{n})^{2}} \left( \frac{X_{i,j+1}^{n} - X_{i,j-1}^{n}}{2\Delta z} \right)^{2} \right)
$$
\n(24)

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### **Heat and Mass Transfer Equation**

The value of each parameter in both heat and mass transfer equations for this study is taken from an experiment conducted by [9] and presented in the table below.

Table 2: Parameters to solve the heat and mass transfer equations

Parameter	Value
Density of the solids, $\rho$	$600 \text{ kg/m}^3$
Oil content of the cookie, Y	$0.5$ kg oil/kg solids
Specific heat, $C$	$2.30 + 4.19 X_{i,j}^{n}$ kJ/kg
Heat transfer coefficient (radial direction), $k_p$	57.9
Heat transfer coefficient (axial direction), $k_z$	28.94
Latent heat of vaporization, $\lambda$	$2263$ kJ/kg
Diffusivity of water in the cookie, $D$	$1e^{-2}m^2/s$

### **RESULTS AND DISCUSSION**

Temperature and moisture profiles based on the heat and mass transfer equations will be solved using the derived model from objective 1. The graph of temperature profile *T* versus radial *r* will demonstrate how the temperature of the cookie changes along the radial distance at different times. Similarly, for the moisture profile, the graph moisture *X* versus radial *r* is plotted, which illustrates the relationship between radial distance and moisture content in the cookie throughout the baking process. To find the solution for objective 2, the FDM algorithm from objective 1 will be developed and solved with the aid of the MATLAB software.

### **Numerical Result**

Given the radial distance of the cookie from the pan, *r* is between 0 to 2.5 cm. Then, set ∆*r =* 0.25, ∆*z =* 0.104. Hence, there will be 11 iterations for both *r* and *t*. For *t,* ∆*t* = 0.5 mins. The iterations for *r, z* and *t* are presented in Table 3. In the central section of the cookie, which has a radial distance between 1.5 cm and 2.5 cm, heat and mass transfer occur through convection for both moisture and heat transfer [9]. However, in the outer section of the cookie, with a radial distance between 0 and 1.5 cm, the heat and mass transfer occur through conduction and diffusion, respectively. As a result, there is no change in the temperature and moisture content in the radial range of  $0.5 \le r \le 2.5$ .

### **Temperature Profile**

Since we assumed that the oven temperature is  $130^{\circ}$ C, the boundary condition for the left and bottom side of the cookie is 130°C, since these both sides touch the pan or baking tray as illustrated in Figure 2.



Figure 2: An illustration of a cookie on the pan

For the heat transfer equation, the values of parameters  $\rho$ , C,  $\lambda$ ,  $k_R$  and  $k_R$  in Table 2 are substituted into Equation (20). Next, the equation is implemented in the MATLAB software and the temperature profile is plotted at different cookie thickness,  $z = 0.208$  cm, 0.624 cm and 1.04 cm, as shown in Figure 3.



Figure 3: Temperature profiles at  $z = 0.208$  cm, 0.624 cm and 1.04 cm

Figure 3 shows the temperature decreases as the radial distance from the side of the pan increases. This happens due to the heat energy diffusing from the pan (at  $r = 0$ ) to the central region of the cookie (at  $r = 2.5$  cm). The red, blue, and green lines indicate  $z = 0.208$  cm, 0.624 cm and 1.04 cm, respectively. At the beginning of the baking process  $(t=0)$ , the temperature of the whole cookie is at room temperature 30 $^{\circ}$ C. Then, at the end of the baking process at  $t = 20$  mins, the temperature of the whole cookie is at the optimum temperature,  $130^{\circ}$ C. At  $r = 0$ , the temperature is  $130^{\circ}$ C, but decreases for 0 cm  $\leq$  *r*  $\leq$  1.5 cm, while remaining constant for 1.5 cm  $\leq$  *r*  $\leq$  2.5 cm due to convection. Furthermore, as the baking time increases, the temperature of the cookie increases, too.

The gap between the line graph at  $r = 5$  min (dotted line) and at  $t = 10$  min (dashed line) is bigger than the gap between the graph at  $t = 10$  min (dashed line) and at  $t = 15$  min (star) for all levels of cookie thickness. Therefore, as the temperature reaches the optimum level (130°C), the temperature increases at a slower rate. Hence, the increment in temperature at the early stage of baking is faster than at the end of the baking process.

#### **Moisture Profile**

For the mass transfer equation, the value of parameter *D* as presented in Table 2 is substituted into Equation (24). The equation is implemented in the MATLAB software to plot the moisture profile at different cookie thickness,  $z = 0.208$  cm, 0.624 cm and 1.04 cm, as shown in Figure 4. The initial moisture was assumed to be 0.6, and therefore  $X_0 = 0.6$ .



Figure 4: The moisture profiles at  $z = 0.208$  cm, 0.624 cm and 1.04 cm

Figure 4 shows the moisture content from  $t = 0$  until  $t = 20$  at different values of z. The red, blue, and green lines indicate  $z = 0.208$  cm,  $0.624$  cm and 1.04 cm, respectively. As the radial distance from the side of the pan increases, the moisture content in the outer region of the cookie (at r between 0 and 1.5 cm) also increases, while it remains unchanged in the central region due to convection. Additionally, as the baking time increases, the moisture content in the cookie decreases. At  $t = 0$ , the moisture content is 0.6, and the moisture content gradually decreases throughout the baking process. At the end of the baking process, the whole cookie does not have a uniform moisture content. The side of the cookie has a lower moisture content than the central region. This indicates that the cookie produced has a crunchy outside and a soft inside.

#### **Temperature versus Moisture Profile**

In Subtopic 3.3, it was discussed how temperature and moisture changes occur at different times. Therefore, Figure 5 will provide a better understanding of how temperature and moisture change at different cookie thickness of the cookie at the same baking time,  $t = 5$  min.



Figure 5: The temperature and moisture profiles at  $z = 0.208$  cm, 0.624 cm and 1.04 cm

Based on Figure 5, we can see the relationship between the moisture content and temperature in the baking process. As the radial distance from the side of the pan increases, the temperature will decrease, but moisture content increases. Therefore, an increase in temperature leads to a reduction

in moisture content. At  $t = 2.5$  cm, the temperature for  $z = 0.208$  cm, 0.624 cm and 1.04 cm are 90°C, 50°C and 38°C, respectively. If the thickness of the cookie, *z*, increases, it means the region is far from the heat source (pan). Hence, when the thickness of the cookie increases, the time taken for the temperature of the cookie to increase increases, too. Therefore, the increment in temperature will be faster when in the regions are closer to the heat source (pan).

# **CONCLUSION**

Investigate heat and moisture transfer in the baking process of cookies. By solving the twodimensional PDE heat and mass transfer equations using the Finite Difference Method (FDM), specifically FTCS, the relationship between radial distance from the side of the pan, temperature, and moisture content was explored. The results showed that as the radial distance from the side of the pan increased, the temperature decreased while the moisture content increased. At the end of the baking process, the entire cookie reached the optimum temperature, but variations in moisture content were observed. The side of the cookie had lower moisture content compared to the inner region, resulting in a crunchy exterior and a soft interior. This study contributes to a better understanding of the baking process and its effects on cookie properties.

In future research, the heat and mass transfer of cookies in different shapes, such as rectangles or stars, could be considered to investigate whether the cookie's shape affects these processes. In this study, the temperature profile is based on the cookie's temperature. Hence, in future, the temperature profile can be plotted based on the cooking equipment's temperature too, such as, the oven and the baking tray.

# **CONFLICTS OF INTEREST**

The authors declare no conflict of interest.

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### **REFERENCES**

- [1] Hirschmann, R. (2022, May 18). Malaysia: Sales value of manufactured bread, cakes and other bakery products 2021. Statista. Retrieved from Sales Value of Manufactured Bakery Products. https://www.statista.com/statistics/642408/sales-value-of-manufactured-bread-cakes-andother-bakery-products-in-malaysia/
- [2] Ureta, M. M., Olivera, D. F., & Salvadori, V. O. (2015). Baking of sponge cake: Experimental characterization and mathematical modelling. *Food and Bioprocess Technology*, *9*(4), 664–674
- [3] Jones, A. Z. (2018, March 26). Introduction to heat transfer: How does heat transfer? *ThoughtCo*. https://www.thoughtco.com/how-does-heat-transfer-2699422
- [4] Lucas, J. (2022, February 28). What is the first law of thermodynamics? *LiveScience*. Retrieved from What Is the First Law Of Thermodynamics https://www.livescience.com/50881-first-lawthermodynamics.html
- [5] Arepally, D., Reddy, R., Goswami, T., & Datta, A. (2020). Biscuit Baking: A Review. *LWT Food Science and Technology*, *131*, 109726. https://doi.org/10.1016/j.lwt.2020.109726
- [6] Silva, T. H., Monteiro, R. L., Salvador, A. A., Laurindo, J. B., & Carciofi, B. A. (2022). Kinetics of bread physical properties in baking depending on actual finely controlled temperature. *Food Control*, *137*, 108898.
- [7] Stych, A. (2021, April 16). Pandemic time off leads to new pastimes. *Bizwomen: The Business Journal*. Retrieved from https://www.bizjournals.com/bizwomen/news/latest-news/2021/04/ pandemic-time-off-leads-to-new-pastimes.html?page=all
- [8] Wang, Y., Chen, L., Yang, T., Ma, Y., McClements, D. J., Ren, F., Tian, Y., & Jin, Z. (2021). A review of structural transformations and properties changes in starch during thermal processing of foods. *Food Hydrocolloids*, *113*, 106543.
- [9] Ozilgen, M., & Heil, J. R. (1994). Mathematical modeling of transient heat and mass transport in a baking biscuit. *Journal of Food Processing and Preservation*, *18*(2), 133–148