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ON (*i, j*) SUPRA R-OPEN SETS AND THEIR APPLICATIONS

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ABSTRACT

This study aims to introduce the (i,j) supra R-open set in supra bitopological spaces as a generalisation of supra R-open sets in supra topological spaces and investigate their main properties. The (i,j) supra R-open sets are introduced and their fundamental properties and applications in supra bitopological spaces are presented. Afterwards, the study explores the notions of (i,j) supra R-continuous, (i,j) supra R^*-continuous and (i,j) supra R-irresolute maps, and investigates their main properties and relationships. After that, the study examines the concepts of (i,j) supra R-open and (i,j) supra R-closed maps and investigates their relationships with other maps, such as (i,j) supra R-continuous (i,j) supra R^*-continuous, (i,j) supra R-irresolute) maps. Finally, the notion of (i,j) supra R-homeomorphism is formulated and descriptions for each type are provided. The main properties of these maps are studied, and their relationships are illustrated through several examples.

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INTRODUCTION

In 1963, Kelly proposed the notion of Bitopological Spaces (BSs) as a generalisation of topological spaces, extending some of the normal separation axioms in a topological space into a BS [12]. Various studies have been dedicated to the study of bitopologies, some of them are discussed here. For instance, Caldas *et al.* [4] introduced and characterised the concepts of β-open sets and their related notions in ideal BSs. Caldas *et al.* [5] also extended the notions of Λ_δ- T_i and Λ_δ- R_i to bitopological spaces for $i = 1, 2$, and defined the notions of pairwise Λ_δ- T_i and Λ_δ- R_i bitopological spaces for $i = 1, 2$. In this context, they studied some of the fundamental properties of such spaces, and their relationship to some other known separation axioms was investigated. Carpitero *et al.* [6] introduced and explored the notions of (i, j)-ω-b-open sets in BSs as a generalisation of (i, j)-b-open sets. They developed its relationship with already defined generalisations of b-open sets, and defined and discussed the properties of (i, j)-ω-b-continuities. Jafari *et al.* [9] presented and studied new separation axioms using (1,2) α-open sets in BSs.

Jelic [10], meanwhile, studied the decomposition of pairwise continuity in BSs, utilising (i, j)-preopen sets to define the decomposition of pairwise continuity and investigate its relationships with other notions. Kılıçman and Salleh [13] introduced two notions of pairwise Lindelöf BSs and demonstrated their properties of them are demonstrated. They further continued their investigation of bitopological separation axioms and also established that pairwise Lindelöf spaces are not a hereditary property. Furthermore, Noiri and Popa [19] introduced the notion of weakly precontinuities in BSs as a generalisation of precontinuities and provided several characterisations. They also derived some properties of weakly precontinuities.

In 1983, Mashhour *et al.* [17] introduced Supra Topological Spaces (STSs) by ignoring the topological intersection conditions, making them more adaptable in describing real-world problems [15] and constructing examples that demonstrate the relationships between various topological concepts. Recently, other researchers also examined supra topological spaces [1, 2, 8]. Al-shami *et al.* [1] introduced new types of limit points and separation axioms on STSs using supra β-open sets. They explored some characterisations and explained the relationships between them through examples, exploring their features and identifying sufficient conditions for equivalent relations. In a separate work, Al-shami *et al.* [2] introduced concepts on STSs using supra semi-open sets and provided some of their characterisations. They introduced a concept of supra semi limit points of a set and studied their main properties, focusing particularly on the spaces with different properties. They also investigated new separation axioms, namely supra semi T_i-spaces (i = 0,1,2,3,4) and provided complete descriptions of each of them. El-Shafei *et al.* [8] defined some concepts on STSs using supra preopen sets and investigated its main properties in detail and demonstrated the implications of these separation axioms among themselves, as well as with ST_i-spaces, using compelling examples.

Open set generalisations are important in topology and supra topology because through generalised open sets mathematicians have been able to generalise the concepts of continuity, density, connectivity, etc. in general topology and studied their properties. Levine [16] introduced semi-open sets in 1963, and Njastad [18] in 1965 with nearly open sets, focusing on the structures of α-sets and β-sets and their classes. Furthermore, El-Shafei *et al.* [7] introduced the concept of supra R-open sets and discussed their properties.

In this paper, as a generalisation of i-open sets in BSs, the notions of (i,j) supra R-open sets in supra bitopological spaces are introduced and explored. Their relationships with already defined generalisations of (i,j) supra open sets in bitopological spaces and supra R-open sets in topological spaces are also developed. Moreover, the properties of supra R-continuous (resp. open, closed, irresolute) maps, (i,j) supra R-homeomorphism are defined and discussed, and the properties of these notions are separately explained.

PRELIMINARIES

In the rest of this paper, we will use the following abbreviations: TS for topological space, STS for supra topological space, BTS for bitopological space, SBTS for supra bitopological space, SO for supra open, SC for supra closed, SRO for supra R-open, SRC for supra R-closed, SCt for supra continuous, iff for "if and only if".

Let X be a non-empty set and τ_1, τ_2 be two topologies on X. Then, for a subset E of X, the interior and closure of E with respect to τ_1 are denoted by τ_1. int(E) and τ_1. cl(E), respectively. A subcollection μ of 2^X is called a supra topology on X [17] if Ø and X belong to μ and μ is closed under unions of arbitrary elements in it. Every element of μ is said to be SO set of (X, μ) and its complement is said to be SC. A subset E of X, the supra interior and supra closure of E, is denoted by $\mathrm{int}_\mu(E)$ and $\mathrm{cl}_\mu(E)$, respectively. If τ is a topology on X and μ is a supra topology on X with $\tau \subseteq \mu$, then, we call μ a supra topology associated with τ.

Let (X, μ) and (Y, ρ) be two STSs. Then, a function f: $(X, \mu) \rightarrow (Y, \rho)$ is said to be SCt if the inverse image of every SO set in Y is SO set in X. The function f is called SO if the image of every SO set in X is SO set in X. Moreover, a bijective function g: $(X, \mu) \rightarrow (Y, \rho)$ is said to be supra homeomorphism if g is both SCt and SO. Let (H, μ) be a STS. A subset E of H is said to be SRO if there exists a nonempty SO set G such that $G \subseteq \mathrm{cl}_\mu(E)$. The complement of the SRO set is called SRC [7]. A subset A of a BS (X,τ_1, τ_2) is said to be (i, j)-α-open [11] (resp. (i, j)-semi-open [3], $(i,$

j)-preopen [10], (*i, j*)-*b*-open [14] if $A \subseteq \tau_i.\text{int}\left(\tau_j.\text{cl}\left(\tau_i.\text{int}(A)\right)\right)$ (resp. $A \subseteq \tau_j.\text{cl}\left(\tau_i.\text{int}(A)\right)$, $A \subseteq \tau_j.\text{int}\left(\tau_i.\text{cl}\left(A\right)\right)$, $A \subseteq \tau_i.\text{cl}\left(\tau_i.\text{int}(A)\right) \cup \tau_i.\text{int}\,\tau_j.\text{cl}\left(A\right)\right)$ where $i, j \in \{1,2\}$, $i \neq j$.

Let (X,τ_1, τ_2) and (Y,θ_1, θ_2) be BSs. Then, a function $f:(X,\tau_1, \tau_2) \rightarrow (Y,\theta_1, \theta_2)$ is said to be *i*-continuous (resp. *i*-open, *i*-closed, *i*-homeomorphism) if the induced functions $f:(X,\tau_i) \rightarrow (Y,\theta_i)$ is continuous (resp. open, closed, homeomorphism) where $i \in \{1,2\}$, see [20, 21, 22].

(*i, j*) SUPRA *R*-OPEN SETS

In this section, the notion of (*i, j*) SRO sets is defined and its relationship with some famous generalised (*i, j*) SO sets is illustrated. Also, its fundamental properties are demonstrated.

Definition 3.1. Let (H, μ_1, μ_2) be an SBTS. A subset E of H is said to be (*i, j*) SRO if there exists $G \in \mu_i - \{\varnothing\}$ such that, $G \subseteq \text{cl}^{\mu_i}(E)$. The complement of (*i, j*) SRO set is said to be (*i, j*) SRC set.

Remark 3.1. The class of all (*i, j*) SRO sets in SBTS (H, μ_1, μ_2) is denoted by (*i, j*) SRO (*H*).

Theorem 3.1. Consider (H, μ_1, μ_2) is a SBTS. A subset B of H is (*i, j*) *SRC* if and only if there is a supra μ_i-closed set $F \neq H$ such that $\text{int}^{\mu_i}(B) \subseteq F$.

Proof. Assume that B is (*i, j*) SRC set. Now, B^c is (*i, j*) SRO set, then, there is $G \in \mu_{i\,i} - \{\varnothing\}$ such that, $G \subseteq \text{cl}^{\mu_i}(B^c)$. Accordingly, $(\text{cl}^{\mu_i}(B^c))^c \subseteq G^c$. Thus, $\text{int}^{\mu_i}(B) \subseteq G^c$. Putting $F = G^c$, we will have $\text{int}^{\mu_i}(B) \subseteq F \neq H$. Conversely, let $B \subseteq H$ and there is (*i, j*) SC set $F \neq H$ such that, $\text{int}^{\mu_i}(B) \subseteq F$, then, $\varnothing \neq F^c \subseteq \left(\text{int}^{\mu_j}(B)\right)^c = \text{cl}^{\mu_i}(B^c)$. Accordingly, B^c is an (*i, j*) SRO set. Hence, B is (*i, j*) SRC set.

Let (H, μ_1, μ_2) be a SBTS. A subset F of H is said to be supra μ_i-open if $F \in \mu_i$ for $i \in \{1,2\}$. The complement of supra μ_i-open set is called supra μ_i-closed set.

Theorem 3.2. Let (H, μ_1, μ_2) be a SBTS. Then every supra μ_i-open set in H is (*i, j*) *SRO* set. Proof. It is clear.

Observe that for any bitopological space (H,τ_1, τ_2) and associated supra bitopological space (H,μ_1, μ_2), every τ_i-open set is (*i, j*) SRO since every τ_i-open set is supra μ_i-open. The converse of Theorem 3.2 is not true as shown by the following counterexample.

Example 3.1. Let $H = \{x,y,z\}$ and $\mu_1 = \{\varnothing, H, \{x,y\}, \{z\}\}$, $\mu_2 = \{\varnothing, H, \{x,y\}\}$ be two supra topologies on H. Then $\{y\}$ is (1,2) SRO set, but not supra μ_1-open set as $\{y\} \notin \mu_1$.

Definition 3.2. A subset A of a SBTS (H,μ_1, μ_2) is called SO if $A \in \mu_1, \mu_2$.

The SO set and (*i, j*) SRO set in an SBTS are independent notions as the examples below show. An SO set is not necessary (*i, j*) SRO as shown below.

Example 3.2. Let $H = \{x,y,z\}$ and $\mu_1 = \{\varnothing, H, \{x,y\}\}$, $\mu_2 = \{\varnothing, H, \{x,y\}, \{z\}\}$ be two supra topologies on H. Then, $\{z\}$ is supra μ_2-open set and hence, supra open, but it is not (1,2) supra *R*-open set since $\text{cl}^{\mu_2}(\{z\}) = \{z\}$ and no nonempty element of μ_1 contained in $\text{cl}^{\mu_2}(\{z\})$.

Likewise, an (*i, j*) supra *R*-open set is not necessary supra open as the example below shows.

Example 3.3. Let $H = \{x,y,z\}$ and $\mu_1 = \{\varnothing, H, \{x,y\}, \{z\}\}$, $\mu_2 = \{\varnothing, H, \{x,y\}\}$, be two supra topologies on H. Then $\{y\}$ is (1,2) SRO set, but not SO open set as $\{y\} \notin \mu_1 \cup \mu_2$.

Definition 3.3. A subset A of an SBTS (H, μ_1, μ_2) is said to be (i, j) supra b-open if $A \subseteq cl^{\mu_i} int^{\mu_j}(A) \cup int^{\mu_i}(A)$ where $i, j \in \{1,2\}$, $i \neq j$.

Theorem 3.3. Every (i,j) supra b-open set in a supra bitopological space (H, μ_1, μ_2) with nonempty μ_i-interior is (i,j) supra R-open set.

Proof. Suppose that E is an (i,j) supra b-open set with nonempty μ_i-interior. Then, $int^{\mu_i}(E) \neq \varnothing$ and $int^{\mu_i}(E) \subseteq E \subseteq cl^{\mu_i} int^{\mu_i}(E) \cup int^{\mu_i} cl^{\mu_j}(E) \subseteq cl^{\mu_i}(E)$. Therefore, E is (i,j) supra R-open set.

The converse of Theorem 3.3 is not true as counterexample below shows.

Example 3.4. Let $H = \{x,y,z\}$ and $\mu_1 = \{\varnothing\ H, \{x\}, \{z\}, \{x,z\}\}$, $\mu_2 = \{\varnothing\ H, \{z\}, \{x,y\}\}$ be two supra topologies on H. Then, $\{y\}$ is (1,2) SRO set, but is not (1,2) supra b-open set.

Proposition 3.1. Every (i,j) supra neighbourhood of any point in a supra bitopological space (H, μ_1, μ_2) is (i,j) SRO set.

Proposition 3.2. If E is (i,j) SRO set in SBTS (H, μ_1, μ_2), then, $E \cup A$ is (i,j) SRO set for any $A \subseteq H$.

Theorem 3.4. Let (H, μ_1, μ_2) be an SBTS. Then, the union of an arbitrary (i,j) SRO sets is (i,j) SRO set.

Proof. Let $\{E_\alpha : \alpha \in I\}$ be a family of (i,j) SRO sets. Then, there exist $\alpha_0 \in I$ and $G \in \mu_i - \{\varnothing\}$ such that, $G \subseteq cl^{\mu_j}(E_{\alpha 0}) \subseteq cl^{\mu_j}(\cup_{\alpha \in I} E_\alpha)$. Hence, $\cup_{\alpha \in I} E_\alpha$ is (i,j) SRO set.

Remark 3.2. The intersection of a finite number of (i,j) SRO sets may not be (i,j) SRO set as shown by the following example.

Example 3.5. Let $H = \{x,y,z\}$ and $\mu_1 = \{\varnothing\ H, \{x\}, \{y\}, \{x,z\}, \{y,z\}\}$, $\mu_2 = \{\varnothing\ H, \{x\}, \{y\}, \{x,y\}\}$ be two supra topologies on H. Now, $\{x,z\}$ and $\{y,z\}$ are (1,2) SRO sets, but $\{x,z\} \cap \{y,z\} = \{z\}$ is not (1,2) SRO set.

Theorem 3.5. Let (H, μ_1, μ_2) be a SBS. Then, the intersection of an arbitrary (i,j) SRC sets is (i,j) SRC set.

Proof. Let $\{B_\alpha : \alpha \in I\}$ be a family of (i,j) SRC sets. Then, $\{B^c_\alpha : \alpha \in I\}$ is a family of (i,j) SRO sets. Therefore, $\cup_{\alpha \in I} B^c_\alpha$ is (i,j) is SRO set by Theorem 3.4 Therefore, $\cap_{\alpha \in I} B_\alpha$ is (i,j) SRC set.

Remark 3.3. The union of a finite number of (i,j) SRC sets may not be (i,j) SRC set as shown by the following example.

Example 3.6. Let $H = \{x,y,z\}$ and $\mu_1 = \mu_2 = \{\varnothing\ H, \{x,y\}, \{y,z\}\}$ be two supra topologies on H. Now, $\{x\}$ and $\{z\}$ are (1,2) SRC sets, but the union of them $\{x,z\}$ is not (1,2) SRC set.

Definition 3.4. The (i,j) supra R-interior of E (denoted by (i,j) int_i R (E)) is the union of all (i,j) SRO sets contained in E. The (i,j) supra R-closure of E (denoted by (i; j) cl_i R (E)) is the intersection of all (i,j) SRC sets containing E.

The following theorems are obvious, so the proof is omitted.

Theorem 3.6. Let (H, μ_1, μ_2) be a supra bitopological space. Then:

(i) $A \subseteq cl_R^{\mu_i}(A)$; and $A = cl_R^{\mu_i}(A)$ *iff* A is (i,j) SRC set.

(ii) $(i,j)\ \text{int}_R^{\mu_i}(A) \subseteq A$; and $A = (i,j)\ \text{int}_R^{\mu_i}(A)$ *iff* A is (i,j)SRO set.

(iii) $H - (i,j)\text{int}_R^{\mu_i}(A) = (i,j)cl_R^{\mu_i}(H - A)$.

(iv) $H - (i,j)\ cl_R^{\mu_i}(A) = (i,j)\ \text{int}_R^{\mu_i}(H - A)$.

Theorem 3.7. Let (H, μ_1, μ_2) be a SBTS. Then:

(i) $(i,j)\ \text{int}_R^{\mu_i}(A) \cup (i,j)\ \text{int}_R^{\mu_i}(B) \subseteq (i,j)\ \text{int}_R^{\mu_i}(A \cup B)$.

(ii) $(i,j)\ cl_R^{\mu_i}(A \cap B) \subseteq (i,j)\ cl_R^{\mu_i}(A) \cap (i,j)\ cl_R^{\mu_i}(B)$.

In the above theorem, the inclusion relation could not be replaced by equality relations as the counterexample below shows:

Example 3.7. Let $H = \{a,b,c\}$ and $\mu_1 = \mu_2 = \{\emptyset, H, \{a,b\}, \{b,c\}\}$ be two supra topologies on H. For $A = \{b\}$ and $B = \{c\}$, then, $(1,2)\ \text{int}_R^{\mu_1}(A) = \{b\}, (1,2)\ \text{int}_R^{\mu_1}(B) = \emptyset$ and $(1,2)\ \text{int}_R^{\mu_1}(A \cup B) = \{b,c\}$. Also, for $C = \{b,c\}$ and $D = \{a,c\}$, then, $(1,2)\ \text{cl}_R^{\mu_1}(C) = H, (1,2)\ \text{cl}_R^{\mu_1}(D) = D$ and $(1,2)\ \text{cl}_R^{\mu_1}(C \cap D) = \{c\}$.

DIFFERENT TYPES OF MAPPING BETWEEN SUPRA BITOPOLOGICAL SPACES

There are many types of mapping between supra topological spaces. So, in this section, three different types of mapping between supra bitopological spaces are discussed. The details of these mappings are presented in the following subsections.

(i.j) supra *R*-continuous Maps

This subsection focuses on introducing the notions of (i, j) supra R-continuous (SRCt), (i, j) supra R^*-continuous (SR^*Ct), and (i, j) supra R-irresolute (SRIr) maps between supra bitopological spaces. Moreover, we determine the relationships between them and investigate their properties.

Definition 4.1. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be BSs, and μ_1, μ_2 (resp. ρ_1, ρ_2) be associated supra topologies with τ_1, τ_2 (resp. θ_1, θ_2). A map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is said to be (i, j) SRCt (resp. (i, j) SR^*Ct, (i, j) SRIr) if the inverse image of each θ_j-open (resp. supra ρ_j-open, (i, j) SRO) subset of Y is (i, j) SRO subset of X.

Theorem 4.1. If g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is an i-continuous map, then, g is also (i, j) SRCt map.

Proof. Assume that g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is a i-continuous map. If E is θ_j-open subset of Y, then, $g^{-1}(E)$ is τ_1-open subset of X. Thus, $g^{-1}(E)$ is (i, j) SRO set, and hence, g is (i, j) SRCt.

The converse of Theorem 4.1 is not true in general as the following counterexample shows:

Example 4.1. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be BSs, and μ_1, μ_2 (resp. ρ_1, ρ_2) be associated supra topologies with τ_1, τ_2 (resp. θ_1, θ_2). Consider $X = \{a, b, c\}$, and $\tau_1 = \{\emptyset, X, \{a\}, \{a,b\}\}$, $\tau_2 = \{\emptyset, X, \{a\}, \{a,c\}\}$, $\mu_1 = \tau_1 \cup \{b\}$, $\mu_2 = \tau_2 \cup \{c\}$, $Y = \{x,y,z\}$, $\theta_1 = \{\emptyset, Y, \{y,z\}\}$, $\theta_2 = \{\emptyset, Y, \{x,z\}\}$ and $\rho_1 = \theta_1 \cup \{x,y\}$, $\rho_2 = \theta_2 \cup \{y,z\}$. Let g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ be a map defined as follows:

$$g: (a) = y,\ g(b) = x \text{ and } g(c) = z.$$

Then, g is (1,2) SRCt, but it is not 1-continuous since $g^{-1}(\{y,z\}) = \{a,c\} \notin \tau_1$ for $\{y,z\} \in \theta_1$. Moreover, g is also (1,2) supra R^*-continuous.

Since each θ_i-open set is supra ρ_i-open, we have:

Theorem 4.2. Each (i, j) supra R^*-continuous is (i, j) supra R-continuous.

The converse of Theorem 4.2 is not true as shown by the following counterexample:

Example 4.2. By Example 4.1, take $\tau_1 = \{\varnothing, X, \{a\}, \{a,c\}\}$, $\mu_1 = \tau_1 \cup \{c\}$, $\tau_2\{\varnothing, X, \{c\}, \{b,c\}\}$, $\mu_2 = \tau_2 \cup \{a,c\}$, $\rho_1 = \theta_1 \cup \{x\}$, θ_1, θ_2 and ρ_2 remain unchanged. Then g is 1-continuous and also (1,2) supra R-continuous, but g is not (1,2) SR*Ct function since $\{x\}$ is supra ρ_1-open set in Y but $g^{-1}(\{x\}) = \{b\}$ is not (1,2) SRO set in X.

Note that, Examples 4.1 and 4.2 show that i-continuity and (i,j) SR*Ct are independent notions.

Theorem 4.3. Every (i,j) *RIr* map is (i,j) *SR*Ct.*

The converse of Theorem 4.3 is not true as the counterexample below shows:

Example 4.3. Let $X = \{a,b,c,d\}$ and $\tau_1 = \tau_2 = \mu_1 = \mu_2 = \{\varnothing, X, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}\}$, $Y = \{x,y,z\}$, $\theta_1 = \theta_2 = \{\varnothing, Y\}$ and $\rho_1 = \rho_2 = \{\varnothing, Y, \{y,z\}, \{x\}\}$. The map $g: (X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)$ defined as $g(a) = g(b) = x$, $g(c) = y$ and $g(d) = z$. Then, g is (1,2) SR*Ct map, but is not (1,2) RIr map since $\{z\}$ is (1,2) SRO in Y and $g^{-1}(\{z\}) = \{d\}$ is not (1,2) SRO in X.

Theorem 4.4. Consider (X, τ_1, τ_2) and (Y, θ_1, θ_2) are BSs. Let $g: (X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)$ be a map and μ_i is an associated supra topology with τ_i. The following statements are equivalent:

(i) g is a (i,j) SRCt map;

(ii) The inverse image of every θ_i-closed subset of Y is a (i,j) SRC subset of X;

(iii) $(i,j)\ cl_R^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}(\theta_i.cl(A))$ for each $A \subseteq Y$;

(iv) $g((i,j)\ cl_R^{\mu_i}(E)) \subseteq \theta_i.cl(g(E))$ for each $E \subseteq X$;

(v) $g^{-1}(\theta_i.\mathrm{int}(A)) \subseteq (i,j)\ \mathrm{int}_R^{\mu_i}(g^{-1}(A))$ for each $A \subseteq Y$.

Proof. (i) $\Rightarrow$ (ii): Suppose that E is θ_i-closed subset of Y. Then, Y-E is θ_i-open subset of Y. Therefore, $g^{-1}(Y\text{-}E) = X - g^{-1}(E)$ is (i,j) SRO subset of X. Hence, $g^{-1}(E)$ is (i,j) SRC subset of X.

(ii) $\Rightarrow$ (iii): For any subset A of Y, $\theta_i.cl(A)$ is a θ_i-closed subset of Y. Since $g^{-1}(\theta_i.\mathrm{cl}(A))$ is an (i,j) SRC subset of X, then, $(i,j)\ cl_R^{\mu_i}(g^{-1}(A)) \subseteq (i,j)\ cl_R^{\mu_i}(g^{-1}(\theta_i.cl(A))) = g^{-1}(\theta_i.cl(A))$.

(iii) $\Rightarrow$ (iv): Let E be any subset of X. Then, $(i,j)\ cl_R^{\mu_i}(E) \subseteq (i,j)\ cl_R^{\mu_i}(g^{-1}(g(E)))$, but $(i,j)\mathrm{cl}_R^{\mu_i}(g^{-1}(g(E))) \subseteq g^{-1}(\theta_i.\mathrm{cl}(g(E))$. Thus, $g((i,j)\mathrm{cl}_R^{\mu_i}(E)) \subseteq g\left(g^{-1}(\theta_i.\mathrm{cl}(g(E)))\right) \subseteq \theta_i.\mathrm{cl}(g(E))$.

(iv) $\Rightarrow$ (v): Let A be any subset of Y. Then $g((i,j)\ \mathrm{cl}_R^{\mu_i}(X\text{-}g^{-1},(A))) \subseteq \theta_i.\mathrm{cl}(g(X\text{-}g^{-1}(A)))$. But $g(X\text{-}(i,j)\ \mathrm{int}_R^{\mu_i}(g^{-1}(A))) = g((i,j)\mathrm{cl}_R^{\mu_i}(X\text{-}g^{-1}(A))) \subseteq \theta_i.\mathrm{cl}(g(X\text{-}g^{-1}(A))) = \theta_i.\mathrm{cl}(g(Y\text{-}A) = Y\text{-}\theta_i.\mathrm{int}(A)$. Thus, $X\text{-}(i,j)\ \mathrm{int}_R^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}(Y\text{-}\theta_i.\mathrm{int}(A)) = X - g^{-1}(\theta_i.\mathrm{int}(A))$. Hence, $g^{-1}(\theta_i.\mathrm{int}(A)) \subseteq (i,j)\mathrm{int}_R^{\mu_i}(g^{-1}(A))$.

(v) $\Rightarrow$ (i): Suppose that A is any θ_i-open subset of Y, then, $A = \theta_i.\text{int}(A)$. Since $g^{-1}(\theta_i.\text{int}(A)) \subseteq (i,j)$ $\text{int}_R^{\mu_i}(g^{-1}(A))$, we must have $g^{-1}(A) \subseteq (i,j)\ \text{int}_R^{\mu_i}(g^{-1}(A))$. Since $(i,j)\text{int}_R^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}(A)$, then, $g^{-1}(A) = (i,j)\ \text{int}_R^{\mu_i} g^{-1}(A))$. Therefore, $g^{-1}(A)$ is (i,j) SRO set in X. Hence, g is (i,j) SRCt map.

Theorem 4.5. Consider (X, μ_1, μ_2) and (Y, σ_1, σ_2) are supra bitopological spaces and let g: $(X, \mu_1, \mu_2) \to (Y, \sigma_1, \sigma_2)$ be a map. The following statements are equivalent:

(i) g is (i,j) SR*Ct map;

(ii) The inverse image of every supra σ_i-closed subset of Y *is* (i,j) SRC subset of X;

(iii) $(i,j)\ \text{cl}_R^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}cl_i^{\sigma}(A)$ for each $A \subseteq Y$;

(iv) $g\big((i,j)\text{cl}_R^{\mu_i}(E)\big) \subseteq cl_i^{\sigma}(g(E))$ for each $E \subseteq X$;

(v) $g^{-1}(\text{int}_i^{\sigma}(A)) \subseteq (i,j)\text{int}_R^{\mu_i}(g^{-1}(A))$ for each $A \subseteq Y$.

Theorem 4.6. Consider (X, μ_1, μ_2) and (Y, σ_1, σ_2) are SBTSs and let g:$(X, \mu_1, \mu_2) \to (Y, \sigma_1, \sigma_2)$ *be a map. The following statements are equivalent:*

(iii) g is (i,j) *SRIr*;

(ii) The inverse image of each (i,j) SRC subset of Y is (i,j) SRC subset of X;

(iii) $(i,j)\ cl_R^{\mu_i} g^{-1}(A)) \subseteq g^{-1}\big((i,j)cl_R^{\mu_i}(A)\big)$ for each $A \subseteq Y$;

(iv) $g\ ((i,j)\ cl_R^{\mu_i}(E)) \subseteq (i,j)\ cl_R^{\sigma_i}(g(E))$ for each $E \subseteq X$;

(v) $g^{-1}\big((i,j)\ \text{int}_R^{\sigma_i}(A)\big) \subseteq (i,j)\ \text{int}_R^{\mu_i}(g^{-1}(A))$ for each $A \subseteq Y$.

Theorem 4.7. Consider (X, μ_1, μ_2) and (Y, σ_1, σ_2) are BSs and let μ_i and σ_i be associated supra topologies with τ_i and θ_i, respectively. If one of the following conditions holds, then g: $(X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)$ is a (i,j) SRCt map.

(i) $g(\tau_i.\ cl(E) \subseteq (i,j)\ cl^{\sigma_i}(g(E))$ for each $E \subseteq X$;

(ii) $\tau_i.\ cl(g\text{-}^1(A) \subseteq g\text{-}^1\ (i,j)\ cl_R^{\sigma_i}(A))$ for each $A \subseteq Y$;

(iii) $g\text{-}^1\ (i,j)\ \text{int}_R^{\sigma_i}(A)) \subseteq \tau_i.\ \text{int}\ (g\text{-}^1(A)$ for each $A \subseteq Y$.

Proof. (i) Clearly that $(i,j)\ cl_R^{\mu}(E) \subseteq \tau_i.\ cl(E)$ for each $E \subseteq X$. If condition (i) is satisfied, then, $g(i,j)$ $cl_R^{\mu_i}(E)) \subseteq g(\tau_i.\text{cl}(E) \subseteq (i,j)\ cl_R^{\sigma_i}(g(E)) \subseteq \theta_i.cl(g(E))$. Thus, g is a (i,j) supra R-continuous map by Theorem 4.4 (iv).

(ii) It is clear that $(i,j)\ cl_R^{\sigma_i}(A) \subseteq \theta_i.\ cl(A)$ for each $A \subseteq Y$. If condition (ii) is satisfied, then, $(i,j)\ \text{cl}_R^{\mu_i}$ $(g^{-1}(A)) \subseteq \tau_i.\ \text{cl}(g^{-1}(A)) \subseteq g^{-1}(i,j)\text{cl}_R^{\sigma_i}(A)) \subseteq g^{-1}(\theta_i.cl(A))$. Thus, g is a (i,j) supra R-continuous map by Theorem 4.4 (iii).

(iii) Clearly that $\theta_i.\text{int}(A)) \subseteq (i,j)\ \text{int}_R^{\sigma_i}(A)$ for each $A \subseteq Y$. If condition (iii) is satisfied, then $g^{-1}(\theta_i.$ $\text{int}(A)) \subseteq g^{-1}\big((i,j)\ \text{int}_R^{\sigma_i}(A)\big) \subseteq \tau_i.\text{int}(g^{-1}(A)) \subseteq (i,j)\ \text{int}_R^{\mu_i}(g^{-1}(A))$. Thus, g is a (i,j) supra R-continuous map by Theorem 4.4 (v).

The converse of Theorem 4.7 is not true for each condition.

Example 4.4. Consider $X = \{a,b,c\}$, τ_i and μ_i are indiscrete topologies on X, $Y=\{x,y,z\}$, $\theta_i = \sigma_i = \{\varnothing, Y, \{z\}\}$ and the map g: $(X, \tau_1, \tau_2) \to (Y, \theta_1, \theta_2)$ defined as $g(a) = x$, $g(b) = y$ and $g(c) = z$. Then, g is a (1,2) supra R-continuous map, but the three conditions mentioned in the theorem above are not satisfied as pointed out below:

(i) Let $E = \{c\}$. Then, $g\,(\tau_1.\mathrm{cl}(E)) = Y$ and $(1,2)\ \mathrm{cl}_R^{\sigma_1}(g(E)) = \{z\}$. Thus, $g\,(\tau_1.\mathrm{cl}(E)) \not\subseteq (1,2)\ \mathrm{cl}_R^{\sigma_1}(g(E))$.

(ii) Let $A = \{y\}$. Then, $(\tau_1.\mathrm{cl}(g^{-1}(A)) = X$ and $g^{-1}\,((1,2)\ \mathrm{cl}_R^{\sigma_1}(A)) = \{b\}$. Thus, $\tau_1.\mathrm{cl}(g^{-1}(A)) \not\subseteq g^{-1}\,(1,2)\ \mathrm{cl}_R^{\sigma_1}(A))$.

(iii) Let $A = \{y,z\}$. Then, $g^{-1}((1,2)\ \mathrm{int}_R^{\sigma_1}(A)) = \{b\}$ and $\tau_1.\mathrm{int}(g^{-1}(A)) = \emptyset$. Thus, $g^{-1}\,((1,2)\ \mathrm{int}_R^{\sigma_1}(A)) \not\subseteq \tau_1.\mathrm{int}(g^{-1}(A))$.

Theorem 4.8. Consider (X, μ_1, μ_2) and (Y, σ_1, σ_2) are supra bitopological spaces. If one of the conditions below holds, then, g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is a (i,j) *supra* R*-continuous map.

(i) $g\,(cl^{\mu_i}(E)) \subseteq (i,j)\ cl_R^{\sigma_i}(g(E))$ for each $E \subseteq X$.

(ii) $cl^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}\,((i,j)cl_R^{\sigma_i}(A))$ for each $A \subseteq Y$.

(iii) $g^{-1}\,((i,j)\ \mathrm{int}_R^{\sigma_i}(A)) \subseteq \mathrm{int}^{\mu_i}(g^{-1}(A))$ for each $A \subseteq Y$.

Theorem 4.9. Consider (X, μ_1, μ_2) and (Y, σ_1, σ_2) are supra bitopological spaces. If one of the following conditions holds, then, g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is a (i,j) R-irresolute map.

(i) $g\,((i,j)\ cl_R^{\mu_i}(E)) \subseteq (i,j)\ cl_R^{\sigma_i}(g(E))$ for each $E \subseteq X$.

(ii) $(i,j)\ cl_R^{\mu_i}(g^{-1}(A)) \subseteq g^{-1}((i,j)\ cl_R^{\sigma_i}(A))$ for each $A \subseteq Y$.

(iii) $g^{-1}((i,j)\ \mathrm{int}_R^{\sigma_i}(A)) \subseteq (i,j)\ \mathrm{int}_R^{\mu_i}(g^{-1}(A))$ for each $A \subseteq Y$.

Theorem 4.10. Let (X, τ_1, τ_2), (Y, θ_1, θ_2) and (Z, σ_1, σ_2) be BSs and τ^*_i *(resp.* θ_i, σ_i) be an associated supra topology with τ_i(*resp.* θ_i, σ_i). If g:$(Y, \theta_1, \theta_2) \rightarrow (Z, \sigma_1, \sigma_2)$ is a i-continuous map and f: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is (i,j) SRCt map, then, $g\circ f$ is a (i,j) SRCt map.

Proof. Let G be a σ_i-open subset of Z. Since g is i-continuous map, then, $g^{-1}(G)$ is a σ_i-open subset of Y. Since f is (i,j) SRCt, then, $(g\circ f)^{-1}\,(G) = f^{1}\,(g^{-1}(G))$ is (i,j) SRO subset of X. Thus, $g\circ f$ is (i,j) SRCt.

Theorem 4.11. Let (X, τ_1, τ_2), (Y, θ_1, θ_2) and (Z, σ_1, σ_2) be bitopological spaces and τ^*_i *(resp.* θ^*_i, σ^*_i) be an associated supra topology with τ_i *(resp.* θ_i, σ_i). If g:$(Y, \theta_1, \theta_2)\rightarrow(Z, \sigma_1, \sigma_2)$ is (i,j) R-irresolute map and f:$(X, \tau_1, \tau_2)\rightarrow(Y, \theta_1, \theta_2)$ is (i,j) supra R-continuous map, then $g\circ f$ is *a* (i,j) SRCt map.

(i,j) supra R-open and (i,j) supra R-closed Maps

The concepts of (i,j) SRO maps and (i,j) SRC maps are introduced and their properties studied. Also, some relationships between (i,j) supra R-open (i,j) supra R-closed maps and (i,j) supra R-continuous (i,j) supra R*-continuous, (i,j) supra R-irresolute maps will be investigated.

Definition 4.2. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be BSs and μ_1 (resp. μ_2) be an associated supra topology with θ_1 (resp. θ_2). A map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is said to be (i,j) SRO (resp. (i,j) SRC) if the image of every τ_i-open (resp.τ_i-closed) subset of X is a (i,j) SRO (resp. (i,j) SRC) subset of Y.

Theorem 4.12. The map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$, where μ_1 (*resp.* μ_2) is associated with supra topology with θ_1 (*resp.* θ_2) *is a* (i,j) SRO if and only if g:$(\tau_i.\mathrm{int}(E)) \subseteq (i,j)\ \mathrm{int}_R^{\mu_i} g((E))$ for each subset E of X.

Proof. (Necessity): Assume that g is (i,j) supra R-open map. Thus, $g(\tau_i.\mathrm{int}(E))$ is a (i,j) supra R-open subset of Y. It is clear that $g(\tau_i.\mathrm{int}(E)) \subseteq g(E)$, and so $g(\tau_i.\mathrm{int}(E)) \subseteq (i,j)\ \mathrm{int}_R^{\mu_i} g((E))$.

(Sufficiency): Let E be a τ_i-open subset of X. Then, $g(E) = g(\tau_i.\mathrm{int}(E)) \subseteq (i,j)\ \mathrm{int}_R^{\mu_i}\,(g(E)) \subseteq g(E)$. Therefore, $g(E) = (i,j)\ \mathrm{int}_R\,(g(E))$. Thus, g is a (i,j) supra R-open map.

Theorem 4.13. The map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ where μ_1 *(resp.* μ_2*)* is associated supra topology with θ_1 *(resp.* θ_2*) is* (i,j) SRC if and only if $(i,j)\ cl_R\ (g(E)) \subseteq g(\tau_i.cl(E))$ for every subset E of X.

Proof. ($\Rightarrow$): Suppose that g is (i,j) supra R-closed map. Since $g(E) \subseteq g(\tau_i.cl(E))$ and $g(\tau_i.cl(E))$ is (i,j) supra R-closed set, then, $(i,j)cl_R\ (g(E)) \subseteq g(\tau_i.\mathrm{cl}(E))$.

($\Leftarrow$): Suppose that E is a τ_i-closed subset of X. Since $g(E) \subseteq (i,j)\ cl_R\ (g(E)) \subseteq g(\tau_i.\mathrm{cl}(E)) = g(E)$, therefore, $g(E)$ is (i,j) SRC set. Thus, g is a (i,j) SRC map.

Theorem 4.14. *For a bijective map* g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$*, where* μ_i *and* σ_i *are associated supra topologies with* τ_i *and* θ_i*, respectively. The following statements are equivalent:*

(i) g is a (i,j) SRO map;

(ii) g is a (i,j) SRC map;

(iii) g^{-1} is a (i,j) SRCt map.

Proof. (i) $\Rightarrow$ (ii): Assume that E is a τ_i-closed subset of X. Thus, X-E is τ_i-open subset of X and $g(X\text{-}E)$ is a (i,j) SRO subset of Y. Since g is a bijective map, then, $g(X\text{-}E) = Y\text{-}g(E)$. Therefore, $g(E)$ is a (i,j) SRC subset of Y. Hence, g is a (i,j) SRC map.

(ii) $\Rightarrow$ (iii): Assume that g is (i,j) SRC map and E is a τ_i-closed subset of X. Thus, $g(E)$ is (i,j) SRC subset of Y. Since g is bijective and a (i,j) SRC map, then, $(g^{-1})^{-1}(E) = g(E)$, and therefore, g^{-1} is a (i,j) supra R-continuous map by Theorem 4.4.

(iii) $\Rightarrow$ (i): Suppose that E is τ_i-open subset of X. Then, $(g^{-1})^{-1}(E)$ is a (i,j) SRO subset of Y since g^{-1} is a (i,j) supra continuous map. Since g^{-1} is bijective then, $(g^{-1})^{-1}(E) = g(E)$, and therefore, g is a (i,j) SRO map.

Definition 4.3. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be BSs and μ_1 (resp.μ_2) be an associated supra topology with θ_1 (resp.θ_2). A map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is said to be supra i-open (resp. supra i-closed) if the image of every τ_i-open (resp. τ_i-closed) subset of X is a supra μ_i-open (resp. supra μ_i-closed) subset of Y.

Theorem 4.15. Let f:$(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ and g:$(Y, \theta_1, \theta_2) \rightarrow (Z, \sigma_1, \sigma_2)$ be two maps and τ^*_i *(resp.* θ^*_i, σ^*_i) be an associated supra topology with τ_i *(resp.* θ_1, σ_i). Then:

(i) If f is *i-Ct,* surjective map and $g{\circ}f$ is (i,j) SRO map, then, g is a (i,j) SRO map.

(ii) If g is (i,j) SRCt, injective map and $g{\circ}f$ is i-open map, then, f is a (i,j) SRO map.

(iii) If g is (i,j) SR*Ct, injective map and $g{\circ}f$ is supra i-open map, then, f is a (i,j) SRO map.

(iv) If g is (i,j) SRIr, injective map and $g{\circ}f$ *is* (i,j) SRO map, then, f is a (i,j) SRO map.

Proof. (i) Let G be a τ_i-open subset of Y. Then, $f^{-1}(G)$ is a τ_i-open subset of X and $(g{\circ}f)(f^{-1}(G))$ is (i,j) supra R-open subset of Z. As f is surjective, then, $(g{\circ}f)(f^{-1}(G)) = g\ (f)(f^{-1}(G))) = g(G)$ and then, g is a (i,j) SRO map.

(ii) Let G be τ_i-open subset of X. Then, $(g{\circ}f)(G)$ is a σ_i-open subset of Z and then, $g^{-1}((g{\circ}f)(G))$ is a (i,j) SRO subset of Y. Since g is injective, $g^{-1}((g{\circ}f)(G)) = (g^{-1}g)(f(G)) = f(G)$ and thus, f is a (i,j) SRO map.

(iii) and (iv) can be proved similarly.

(i,j) **supra** ***R*****-homeomorphism Maps**

In this subsection, the notion of (i,j) supra R-homeomorphism maps will be introduced and their fundamental basic properties are studied.

Definition 4.4. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be two BSs and μ_1 (resp. μ_1) and υ_1 (resp. υ_2) be associated supra topologies with τ_1 (resp. τ_2) and θ_1 (resp. θ_2). A bijective map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is said to be (i,j) supra R-homeomorphism if g is both (i,j) supra R-continuous and (i,j) supra R-open.

Let (X,μ_1,μ_2) and (Y,ρ_1,ρ_2) be supra bitopological spaces. The function f: $(X,\mu_1,\mu_2) \rightarrow (Y,\rho_1,\rho_2)$ is called supra i-continuous (resp. supra i-open, supra i-closed, supra i-homeomorphism) if the induced function f: $(X,\mu_i) \rightarrow (Y,\rho_i)$ are supra continuous (resp. supra open, supra closed, supra homeomorphism) where $i \in \{1,2\}$. Since every SO open set is SRO in supra topological spaces, then every supra μ_i-open set is (i,j) SRO in supra bitopological spaces. This would imply that every supra i-homeomorphism map is (i,j) supra R-homeomorphism in supra bitopological spaces. But the converse is not always true as the following counterexamples illustrate.

Example 4.5. By Example 4.1, take $\tau_1 = \{\emptyset, X, \{a,b\}\}$, $\mu_1 = \tau_1 \cup \{b\}$, and τ_2, μ_2, θ_1, ρ_1, θ_2 and ρ_2 remain unchanged. Then, g is (1,2) supra R-continuous, bijective and a (1,2) supra R-open map. So, g is (1,2) supra R-homeomorphism map. But, g is not a supra 1-homeomorphism as $\{b\}$ is a supra μ_1-open set and $g(\{b\}) = \{x\}$ is not a supra ρ_1-open set.

Example 4.6. Let $\mathbb{R}$ be the real line and consider $\tau_1 = \tau_2 = \{\emptyset, \mathbb{R}, E_a = (-\infty,a): a \in \mathbb{R}\}$ the left-hand topology on $\mathbb{R}$, the usual topology μ_i be the associated supra topology with τ_i and let the cofinite topology C_i be the associated supra topology with itself on $\mathbb{R}$. The identity map g: $(\mathbb{R}, \tau_1, \tau_2) \rightarrow (\mathbb{R}, C_1, C_2)$ is (i,j) supra R-homeomorphism, but is not supra 1-homeomorphism, as $(-\infty,10)$ is supra μ_1-open set, but $g((-\infty,10)) = (-\infty,10)$ is not supra C_1-open set.

Theorem 4.16. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be bitopological spaces and μ_i and v_i be associated supra topologies with τ_i and θ_i, respectively. Consider g:$(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ is a bijective and (i,j) SRCt map. Then the following statements are equivalent:

(i) g is (i,j) SRHom;

(ii) g^{-1} is (i,j) SRCt;

(iii) *g is* (i,j) SRC.

Theorem 4.17. Let (X, τ_1, τ_2) and (Y, θ_1, θ_2) be BSs and μ_1 *(resp.* μ_2*)* and v_1 (*resp.* v_2) be associated supra topologies with τ_1 *(resp.* τ_2*)* and θ_1 (*resp.* θ_2). A bijective map g: $(X, \tau_1, \tau_2) \rightarrow (Y, \theta_1, \theta_2)$ *is* (i,j) supra R-homeomorphism if and only if $g(\tau_i\text{-int}(E)) \subseteq (i,j)\ \text{int}_R^{v_i}(g(E))$ and $(i,j)\ cl_R^{v_i}g(E)) \subseteq g(\tau_i\text{-}cl(E))$ for any $E \subseteq X$.

Proof. The map g is (i,j) supra R-homeomorphism if and only if g^{-1} is (i,j) supra R-continuous by Theorem 4.16 if and only if $g(\tau_i\text{-int}(E)) \subseteq (i,j)\ \text{int}_R^{v_i}(g(E))$ and $(i,j)\ cl_R^{v_i}(g(E)) \subseteq g(\tau_i\text{-}cl(E))$ by Theorem 4.4 (v) and (iii), and the fact that g is bijective implies $(g^{-1})^{-1} = g$.

CONCLUSION

This paper began by presenting the notion of (i,j) supra R-open sets and investigating their main properties. Also, we have introduced several different types of mappings between supra bitopological spaces, such as (i,j) supra R-continuous maps, (i,j) supra R-open, and (i,j) supra R-closed maps, and studied their properties and relationships. Finally, this study introduced the notions of (i,j) supra R-homeomorphism maps and discussed their basic properties. Several further investigations can be carried out for future open problems. Some of them are listed below:

(1) Introduce weak and stronger forms of (i,j) supra R-open sets in supra bitopological spaces.

(2) Apply (i,j) supra R-open sets to introduce and study new separation axioms in supra bitopological spaces.

(3) Investigate the chances of applying these notions to information systems, engineering, decision-making, etc.

CONFLICTS OF INTEREST

The authors declare that there is no conflict of interest.

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