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A STUDY OF SOME PROPERTIES FOR CELLULAR-LINDELÖF TOPOLOGICAL SPACES

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ABSTRACT

A topological space T is called cellular-Lindelöf if, for every family $\mathcal{F}$ of pairwise disjoint nonempty open subsets of T, there exists a Lindelöf subspace $L \subseteq T$ such that $F \cap L \neq \emptyset$ for every $F \in \mathcal{F}$. This paper aims to investigate the properties of cellular Lindelöf spaces and their relation with other spaces and find cardinal inequality for cellular-Lindelöf spaces based on the Erdos and Rado theorem. The concept of cellular-Lindelöf was utilised to show the properties of cellular-Lindelöf spaces and their relation with other spaces. We obtain a few examples of topological spaces which are not cellular-Lindelöf. Erdos and Rado theorem is a theorem based on intersecting set families. Cardinal inequality was found based on the lemma from Erdos and Rado theorem and information obtained from some journal papers. The study of cellular-Lindelöf spaces is the extension study of Lindelöf spaces and this is important as it can become a reference material for future researchers. The central object of the study of topological dynamics is a topological dynamical system where topological spaces are needed. Thus, the study of cellular-Lindelöf spaces is also important to the field of dynamical systems.

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INTRODUCTION

Most of the topological spaces in this paper are assumed to be Hausdorff spaces or T_2-spaces. It will then be stated or mentioned clearly if the spaces are not Hausdorff. What are topological spaces and Hausdorff spaces? The most common definition of topological spaces is the definition through open sets. Based on Padledweska and Darmochwal [13], a topological space is an ordered pair (Y, τ) where is a set and τ is a collection of subsets of Y if it satisfies three axioms. Firstly, the empty set and Y itself must belong to τ. Next, the finite intersection of members of τ belongs to τ, and the arbitrary union of members of τ belongs to τ. All elements that belong to τ are considered to be an open set. A set can be given discrete topology if all subsets are open. Some examples of topological spaces are metric spaces, uniform spaces, Cauchy spaces and others. Cauchy spaces were introduced by H.H. Keller in 1968. It comes from the concept of the Cauchy filter, which was to study the completeness in topological spaces.

In the beginning, it was mentioned that all topological spaces are assumed to be Hausdorff spaces. Now, let us look at what the Hausdorff space is. Space M is a T_2-space or Hausdorff space if and only if it satisfies the T_2 axioms, that is, for every x and y that are elements of the space M, $x \neq y$, there exists disjoint open sets A and B in M such that $x \in A$ and $y \in B$. An example of Hausdorff space is the slit disc topology on $\mathbb{R}^2$. This topology is Hausdorff space but not regular space or T_3-

space. Figure 1 can help us to understand further the concept of Hausdorff spaces, M and the sets A and B are open sets.

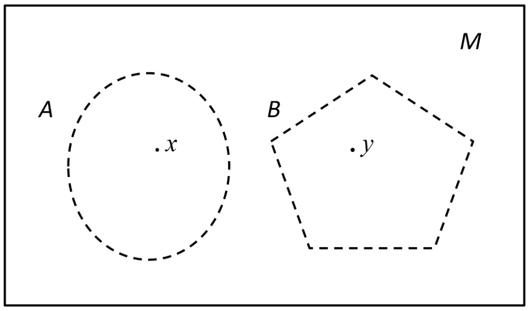

Figure 1: Hausdorff spaces

Compact spaces are the subsets of topological spaces. Compact spaces are defined differently based on different resources. Compactness generalises subset of Euclidean space that is closed and bounded. Based on Darmochwal [6], others such as Bolzano-Weierstrass theorem said that a subset of Euclidean spaces is sequentially compact if closed and bounded. In the study of real analysis, there was a theorem called the Heine Borel theorem. This theorem said that for a compact subset M of Euclidean space, there are two conditions must be fulfilled. First of all, subset M is closed and bounded. Secondly, M is compact if and only if every open cover of M has a finite subcover. A cover of Y is a collection $\mathcal{U}$ of subsets of Y whose union is Y. We had known the concept of compact sets based on the Heine Borel theorem from Hindman [8] and the Bolzano Weierstrass theorem from Morgan [11]. We know that if an open cover of a space has a finite subcover, the space is called compact. Have you ever wondered what if the open cover has a countable subcover? What will that particular topological space be named?

Well, the Finnish mathematician Ernst Leonard Lindelöf discovered this and named it Lindelöf spaces. Cammaroto and Santoro [4] state that Lindelöf space is a topological space where every open cover has a countable subcover. Lindelöf spaces have a close relationship with compact spaces as they require a countable subcover instead of a finite subcover and the countable subcover exists if the particular subcover is finite. An example is the Euclidean spaces. Euclidean spaces are both Lindelöf and separable spaces. Now, have you wondered if there is a space that uses Lindelöf as a subspace? Well, there is and it was called cellular-Lindelöf space.

In 2017, a class of cellular-Lindelöf was introduced and studied by Bella and Spadaro [3]. A topological space T is called cellular-Lindelöf [3] if for every family $\mathcal{F}$ of pairwise disjoint nonempty open sets in T, there exists a Lindelöf subspace $L \subseteq T$ such that $F \cap L \neq \emptyset$ for every $F \in \mathcal{F}$. It all started when Arhangel'skii [1] proved his theorem by saying that every first countable Lindelöf Hausdorff space has cardinality at most c (continuum). Juhász *et al.* [10] introduced the concept of almost discrete Lindelöf space. Juhász also proves that every first countable almost discretely Lindelöf space has the cardinality at most 2^c but is unsure whether the continuum bounds it. Then, based on Arhangel'skii [1], the cardinality of a first countable Hausdorff space with a Souslin number does not exceed the continuum. Later on, Bella and Spadaro [3] proved that almost discretely, Lindelöf first countable space has cardinality at most 2^c that is consistent with Zermelo-Fraenkel set theory (ZFC). Bella and Spadaro [3] also discovered the class of cellular-Lindelöf. They found that the cardinality of cellular-Lindelöf first countable Hausdorff space is at most 2^c while its cellularity is at most c but still unsure whether the cardinality is at most continuum. In this paper, we investigate the properties of cellular-Lindelöf spaces. Next, we are going to find out what is the relation of cellular-Lindelof spaces with other spaces. Finally, we find cardinal inequalities for cellular-Lindelöf using the Erdos-Rado theorem. Paul Erdos, Chao Ko, and Richard Rado basically founded Erdos-Rado theorem. This theorem is based on intersecting set families.

Throughout this paper, by a space, we mean a topological space unless otherwise stated. Definitions and meanings of the symbols will be explained and described in detail for smooth understanding later on.

1. $[Y]$ represents the cardinality of set Y.
2. $[Y]^2$ represents the set of two element subsets of Y.
3. ω be the first infinite cardinal.
4. c be the cardinality of the continuum.
5. The extent of space Y [14], represented by $e(Y)$ is the supremum of the cardinalities of closed discrete subsets of Y.
6. Let Y be a Hausdorff space. The Hausdorff pseudo character of Y [9], denoted by $H_\psi(Y)$ is the smallest infinite cardinal v where for each $x \in X$, there is a collection $\{V(n,x): n < v\}$ of open neighbourhoods of x satisfies the condition if x is not equal to y, then, there exists $m,n < v$ such that $V(m,x) \cap V(n,y) = \emptyset$.
7. A space is separable if and only if it has a countable dense subset.

PROPERTIES OF CELLULAR-LINDELÖF SPACES

In this section, we discuss the properties of cellular-Lindelöf spaces and their relation with other spaces. We shall use the concept and definition of cellular-Lindelöf to prove the properties of cellular-Lindelöf spaces.

Theorem 2.1. If Y is a cellular-Lindelöf space, then any clopen subset of Y is cellular-Lindelöf.

Proof. Let Z be a clopen subset of Y and $\mathcal{U} = \{U_i : i \in \Delta\}$ be a family of pairwise disjoint nonempty open subsets of Z. Since Z is open in Y, then each U_i is also open in Y. We know that Y is a cellular-Lindelöf space, thus, there is a Lindelöf subspace $L \subseteq Y$ such that $L \cap U_i \neq \emptyset$ for each $i \in \Delta$. The set $L \cap Z$ is closed in L since Z is a closed subset of Y. Hence, we can say that the intersection of L and Z is Lindelöf subspace. It can also be seen that $L \cap Z \cap U_i \neq \emptyset$ for each $i \in \Delta$. Therefore, the space Z is cellular-Lindelöf.■

Theorem 2.2. Let $f: Y \rightarrow Z$ be a continuous surjection. If Y is cellular-Lindelöf, then, Z is cellular-Lindelöf.

Proof. Assume that Y is a cellular-Lindelöf space. If $\mathcal{U} = \{U_i : i \in \Delta\}$ is a family of pairwise disjoint nonempty open subsets of z, then, $\mathcal{V} = \{f^{-1}(U_i) : i \in \Delta\}$ is a family of pairwise disjoint nonempty open subsets of Y since $f: Y \rightarrow Z$ is a continuous function. Since Y is cellular-Lindelöf, thus, there exists a Lindelöf subspace $L \subseteq Y$ such that $L \cap f^{-1}(U_i) \neq \emptyset$ for each $i \in \Delta$. We know that a continuous image of a Lindelöf space is Lindelöf, so, we can say that the function that maps Lindelöf subspace L of Y, i.e., $f(L) \subseteq Z$ is Lindelöf, and the intersection of $f(L)$ with U_i is not an empty set for each $i \in \Delta$. This fulfils the condition for Z to be cellular-Lindelöf space. Thus, Z is a cellular-Lindelöf space. ■

From here, we can say that a continuous image of cellular-Lindelöf space is cellular Lindelöf.

Theorem 2.3. If $Y = U_{\kappa \in \omega} Z_\kappa$ where each Z_κ is a cellular-Lindelöf space and ω is the first infinite cardinal, then, Y is cellular-Lindelöf.

Proof. Let $\mathcal{U} = \{U_i : i \in \Delta\}$ be a family of disjoint nonempty open subsets of Y. Assume that $\mathcal{U}_n = \{Z_n \cap U_i : i \in \Delta_n\}$ is a family of nonempty open subsets of Z_n where $\Delta_n \subseteq \Delta$ for each $n \in \omega$. It is clear that each $\mathcal{U}_n$ is disjoint. Thus, there is a Lindelöf subspace $L_n \subseteq Z_n$ such that $L_n \cap Z_n \cap U_i \neq \emptyset$ for all $i \in \Delta_n$. For now, let $L = \bigcup_{n\in\omega} L_n$, so $L \subseteq \bigcup_{n\in\omega} Z_n$. It is obvious that $L \subseteq Y$ is Lindelöf too and the intersection of L and U_i is not empty for all $i \in \Delta$. Thus, Y is cellular-Lindelöf. ■

RELATION OF CELLULAR-LINDELÖF SPACES WITH OTHER SPACES

In this section, we shall utilise the concept of cellular-Lindelöf spaces to determine whether the particular space is cellular-Lindelöf. We will also define the other spaces and their relation with cellular-Lindelöf spaces. Recall that an uncountable space is called an infinite set if the space has too many elements to be countable. A set is uncountable if its cardinality is larger than the cardinality of a natural number, i.e., $\aleph_0$.

Theorem 3.1. Any uncountable space Y with discrete topology is not cellular-Lindelöf.

Proof. Let $\mathcal{U} = \{\{y\} y \in Y\}$. It is clear that $\mathcal{U}$ is a family of pairwise disjoint nonempty open subsets of Y. If the space Y is cellular-Lindelöf, then, there exists a Lindelöf subspace L of Y such that $\{y\} \cap L \neq \emptyset$ for all $y \in Y$. However, it is obvious that $L = Y$. However, the space Y is not Lindelöf since $\mathcal{U}$ is an open cover of Y with no countable subfamily covers Y. This is a contradiction, so, Y is not cellular-Lindelöf.

Example 3.1. The space S x S is separable (and thus, cellular-Lindelöf but not necessarily Lindelöf), where S is the Sorgenfrey line. But S x S has an uncountable closed discrete subspace $L= \{(x,-x): x \in S\} \subset S$ x S which is not cellular-Lindelöf by Theorem 3.1.

The Sorgenfrey plane consists of a product of two copies of the Sorgenfrey line. Based on William G. Bade [2], the Sorgenfrey line is the real line R under the right half-open interval topology. These Sorgenfrey planes and Sorgenfrey lines were named after American mathematician Robert Sorgenfrey. The basis of the Sorgenfrey plane, S x S is the set of rectangles that includes the west edge, southwest corner, and south edge. The basis of the Sorgenfrey plane is a product of Lindelöf spaces but not a Lindelöf space itself. This is because the space S x S has a closed and uncountable discrete subspace. The antidiagonal L which its diagram and equation shown in Figure 2 below is an uncountable discrete subspace of the space S x S. Thus, the space S x S cannot be cellular-Lindelöf by Theorem 3.1.

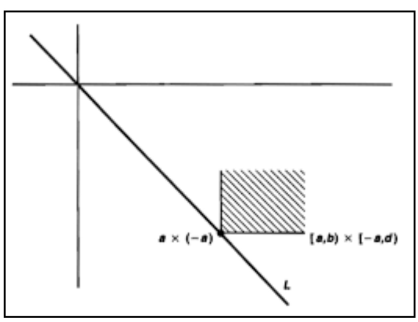

Figure 2: Sorgenfrey plane (The antidiagonal and open rectangle on the right half-open interval in the Sorgenfrey plane that meets the antidiagonal at a single point)

Definition 3.1 [7]. A space X is monotonically normal if there exists a function g which assigns to each ordered pair (H,K) of disjoint closed subsets of X an open set $g(H,K)$ such that:

(a) $H \subseteq g\,(H,K) \subseteq cl\,\big(g\,(H,K)\big) \subseteq X \setminus K$;

(b) If (H',K') is a pair of disjoint closed sets having $H \subseteq H'$ and $K \supseteq K'$, then, $g\,(H,K) \subseteq g\,(H',K')$.

The function g is called a monotone normality operator for X.

Definition 3.2 [7]. A space (X,τ) is semistratifiable if there is a function $S{:}\,\mathcal{H} \times \mathbb{N} \to \tau$ where $\mathcal{H}$ is a family of closed subsets of X, called a semistrafication of X such that:

(a) If $H \subseteq K$ are closed subsets of X, then $S\,(H,n) \subseteq S\,(K,n)$ for every $n \in \mathbb{N}$;

(b) $H = \bigcap_{n\in\mathbb{N}} SH,n$ for every closed set $H \subseteq X$.

If, in addition, the function S satisfies:

(c) $H = \bigcap_{n\in\mathbb{N}} clSH,\, n$ for every closed set $H \subseteq X$

Then, X is stratifiable and S is called a stratification of X.

Theorem 3.3 [7]. A space X is stratifiable if and only if X is semistratifiable and monotonically normal.

In topology, a space is called topologically stratified if it can break up into pieces called strata. Therefore, space Y has a countable extent. This term was introduced by Rene Thom. He showed that every Whitney stratified space was topologically stratified with the same strata. Monotonically normal spaces are a special types of normal spaces with their characteristics. These spaces are hereditarily normal and any two subsets contained in them are strongly separated.

A space Z is called collectionwise normal [7] if for every discrete family $\mathcal{R}$ of closed subsets of Z, there exists a disjoint family $S = \{S(R){:}\ R \in \mathcal{R}\}$ of open subsets of Z such that $R \subseteq S(R)$ for every $R \in \mathcal{R}$. It can be seen that S may be considered a discrete family of open sets.

Theorem 3.4 [14]. If Y is a collectionwise normal and cellular-Lindelöf space, then, $e(Y)$ is countable.

Theorem 3.5 [7]. Any monotonically normal space is (hereditarily) collectionwise normal.

Corollary 3.1 [14]. If Y is a monotonically normal and cellular-Lindelöf space, then, $e(Y)$ is countable.

Proof. Since Y is monotonically normal, then, Y is (hereditarily) collectionwise normal by Theorem 3.5. Thus, $e(Y)$ is countable by Theorem 3.4.■

Theorem 3.5 [5]. Every semistratifiable space which has a countable extent is Lindelöf.

Theorem 3.6. If Y is a cellular-Lindelöf stratifiable space, then, it is Lindelöf.

Proof. Since Y is stratifiable space, then, by Theorem 3.3, Y is semistratifiable and monotonically normal. Hence, $e(Y)$ is countable by Corollary 3.1. So, by Theorem 3.5, we can say that Y is Lindelöf space.■

Manifolds with boundary (top dimension and co-dimension 1 boundary) exemplify stratified space. Manifolds are topological spaces that look locally like Euclidean spaces near each point. One dimensional manifold includes lines and circles but not figure eight, as in Figure 3 below. The picture in Figure 3 shows the lemniscates of Bernoulli, a quartic plane curve. Lemniscates are one of figure eight or ∞-shaped curves. A quartic plane curve is an algebraic plane curve of the fourth degree. For example, a sphere can behave like a circle which can be defined as a subset of $\mathbb{R}^3$. The sphere is two-dimensional and then each part chart will map part of the sphere to an open subset of $\mathbb{R}^2$.

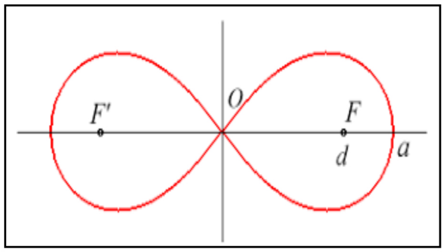

Figure 3: Resources: Lemniscate of Bernoulli (n.d.)
Mathcurvehttps://mathcurve.com/courbes2d.gb/lemniscate/lemniscate.shtml

Based on Micheal [11], if $\mathcal{U}$ is a cover of a topological space X, then, by a refinement of $\mathcal{U}$ we mean a cover $\mathcal{V}$ of X such that every element of $\mathcal{V}$ is a subset of some element of $\mathcal{U}$. A family $\mathcal{G}$ of subsets of X is locally finite if every $x \in X$ has a neighbourhood which intersects only finitely many elements of $\mathcal{G}$. Moreover, a paracompact space is a topological space where every of its open cover has a locally finite open refinement. Note that every compact space is paracompact.

Theorem 3.7. If Y is a cellular-Lindelöf paracompact space, then, it is Lindelöf.

Proof. Since a paracompact space is collectionwise normal, the extent of Y is countable. Now, since every paracompact space with countable extent is Lindelöf, so, it is proven that Y is Lindelöf.■

A STUDY OF CELLULAR-LINDELÖF SPACES

In this section, we find the cardinal inequality for cellular-Lindelöf spaces. Lemma from Erdos Rado theorem was utilised to find the cardinal inequality of cellular-Lindelöf spaces. A few terms that will be used will be defined below. Erdos-Rado theorem is the basic result in partition calculus that is a part of combinatorial set theory. This theorem is the extension of Ramsey's theorem to uncountable sets.

For this section, we will use the lemma from Erdos and Rado theorem to find cardinal inequalities for cellular-Lindelöf spaces. The following lemma is due to Erdos and Rado theorem.

Lemma 4.1 [11, p. 8]. Let ν be an infinite cardinal, Y be a set and $|Y| > 2^\nu$. Suppose that $|Y|^2 = \cup_{\alpha<\nu} P_\alpha$. Then, there exists $\alpha<\nu$ and a subset S of Y with $|S| > \nu$ such that $[S]^2 \subseteq P_\alpha$.

Recall that a space Y is normal if and only if every pair P, Q of closed and disjoint subsets of Y have disjoint open neighbourhoods containing P and Q, respectively. Furthermore, all compact Hausdorff spaces are normal.

Theorem 4.1. If Y is a normal and cellular-Lindelöf space with $H_\psi(Y) = \omega$, then, the space Y has an extent at most c.

Proof. We know that $H_\psi(Y) = \omega$. So, by definition, there is a collection $\{V(n,x):n \in \omega\}$ of open neighbourhoods of x satisfies the condition if x is not equal to y, then, there exists $n,m \in \omega$ such that $V(n,x) \cap V(m,y) = \emptyset$. Now, let us assume that the theorem is false, i.e., the extent of the space Y, $e(Y)$

is larger than c. Thus, there is a closed discrete subset S of Y such that $|S| > c$. For every $n,m \in \omega$, let:

$$P_{n,m} = \{\{x,y\} \in [Y]^2 : V(n,x) \cap V(m,y) = \emptyset\}.$$

It is also clear that:

$$[S]^2 = \bigcup_{n,m,k \in \omega} P_{n,m,k}.$$

Using Lemma 4.1, there is an uncountable subset R of S such that $[S]^2 \subseteq P_{n0,m0}$ for some $n_0, m_0 \in \omega$. Since the subset S is closed and discrete in Y, then, R as a subset of S is closed and discrete in Y too. Since $[R]^2 \subseteq P_{n0,m0}$, then $V(n_0,x) \cap V(m_0,y) = \emptyset$ for every points $x, y \in R, x \neq y$. Now, let $\mathcal{F}$ be a family $\{V(n_0,x) \cap V(m_0,x) : x \in R\}$. This family is a disjoint and open expansion of R. Since Y is normal, thus, there exists an open subset O of Y such that $R \subseteq O \subseteq cl(O) \subseteq \cup \mathcal{F}$.

Now, let $V(x) = V(n_0,x) \cap V(m_0,x) \cap O$ for all $x \in R$. One can easily find that $\mathcal{V} = \{V(x): x \in R\}$ is a family of nonempty open subsets of Y that are discrete (hence disjoint). The space Y is cellular-Lindelöf, so, there is a Lindelöf subspace L of Y such that $V(x) \cap L \neq \emptyset$ for each $V(x) \in V$. For every $x \in R$, take $q_x \in V(x) \cap L$ and let $C = \{q_x : x \in R\}$. From here, we can conclude that C is an uncountable closed and discrete subset of L. This contradicts the fact that L is Lindelöf. This indirectly proves that extent of the space Y is at most continuum, which is c. This completes the proof. ■

CONCLUSION

Cellular-Lindelöf spaces did utilise Lindelöf spaces to determine whether a topological space is cellular-Lindelöf. Based on the results that had been found, some properties of cellular-Lindelöf are a clopen subset of cellular-Lindelöf is cellular-Lindelöf, the continuous image of cellular-Lindelöf is cellular Lindelöf and union of cellular-Lindelöf is cellular-Lindelöf. As suggestions for future research, one can find and research more on cardinal inequality for cellular-Lindelöf spaces. The cardinal inequality for cellular-Lindelöf in this paper was based on the lemma from the Erdos-Rado theorem. The next researcher can find cardinal inequality for cellular-Lindelöf spaces with the help of other theorems other than the Erdos-Rado theorem. Besides that, one can also discover and research more on properties of cellular-Lindelöf spaces and their relation with other spaces.

This research contributes a lot to humanity and the field of mathematics study. The study of dynamical systems can help us to understand oxygen and carbon dioxide transport, enzyme kinetics and many more in biology. With the knowledge of biology, one can figure and solve problems or handle diseases that affect our bodies using the medical treatment and others. This indirectly provides much help to humanity.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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