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A CONNECTION BETWEEN RICKART REAL C*-ALGEBRA AND ENVELOPING RICKART C*-ALGEBRA

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ABSTRACT

In the paper, Rickart complex and real C*-algebra are considered. For Rickart's real C*-algebra, its connection with the enveloping (complex) C*-algebra is studied. It is shown that the fact that A is a Rickart real C*-algebra does not imply that a complexification A +i A of A is a Rickart (complex) C*-algebra. Proved that if A is a real C*-algebra and A +i A is a Rickart C*-algebra, then, A +i A is a Rickart real C*-algebra. It is shown that there exists a Rickart real C*-algebra whose projection lattice is not complete.

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INTRODUCTION

The theory of operator algebras was initiated in a series of papers by Murray and von Neumann in the thirties. Later such algebras were called von Neumann algebras or W*-algebras. These algebras are self-adjoint unital subalgebras M of the algebra $B(H)$ of bounded linear operators on a complex Hilbert space H, which are closed in the weak operator topology. Equivalently, M is a von Neumann algebra in $B(H)$ if it is equal to its commutant (von Neumann's bicommutant theorem). A factor (or W*-factor) is a von Neumann algebra with a trivial centre and investigation of general W*-algebras can be reduced to the case of W*-factors, which are classified into types I, II and III. The study of C*-algebras (self-adjoint subalgebras in $B(H)$ which are closed in the norm topology) was begun in the work of Gelfand and Naimark, who proved that such algebras could be characterized abstractly as Banach *-algebras satisfying conditions connecting the norm and the involution. They also proved the fundamental result (Gelfand-Naimark theorem) that a commutative initial C*-algebra is isomorphic to the algebra of complex-valued continuous functions on a compact space (its spectrum). Nowadays, the theory of W*-algebras and C*-algebras is a deeply and widely developed theory interacting with many branches of mathematics and several areas of theoretical physics.

Rings and algebras, which will be discussed below were first studied by C.E. Rickart [1]. The main properties of these algebras (i.e., Rickart C*-algebras) are given in Berberyan's monograph [2]. We will also consider some of them below. These algebras were further developed in the works of I. Kaplansky [3-5]. Exactly, AW*-algebras was proposed by I. Kaplansky as an appropriate setting for certain parts of the algebraic theory of von Neumann algebras. Since AW*-algebras are a generalization of Rickart C*-algebras and von Neumann algebras, the theory of AW*-algebras began to develop faster. Rickard C*-algebras played an essential role in the emergence of AW*-algebras.

In this article, we consider the real analogue of these algebras. Namely, in the paper, Rickart complex and real C*-algebra are considered. For Rickart's real C*-algebra, its connection with the enveloping (complex) C*-algebra is studied. It is shown that the fact that A is a Rickart real C*-algebra does not imply that a complexification A +i A of A is a Rickart (complex) C*-algebra. Proved that if A is a real C*-algebra and A +i A is a Rickart C*-algebra, then, A is a Rickart real C*-algebra. It is shown that there exists a Rickart real C*-algebra whose projection lattice is not complete.

PRELIMINARIES

Definition 2.1. A *C*-algebra* is a (complex) Banach *-algebra whose norm satisfies the identity $\|x^*x\|=\|x\|^2$. Now, let A be a real Banach *-algebra. A is called a *real C*-algebra* if $A_c = A+iA$ can be normed to become a (complex) C*-algebra by extending the original norm on A.

Note that a C*-norm on A_c is unique if it exists. It is known that [6, Corollary 5.2.11] A is real C*-algebra if and only if $\|x^*x\|=\|x\|^2$ and $1+x^*x$ is invertible for any $x \in A$. Let us show the importance of the last condition, i.e., the essentiality of element reversibility $1+x^*x$. Consider the algebra $A=\mathbb{C}$ as a real Banach *-algebra with the norm $\|a\| = |a|$ and involution $a^* = a \forall a \in \mathbb{C}$. It is easy to see that $\|aa^*\|=\|a\|^2$, for any $a \in A$. However, for the element $a= i$ we have $1+i \cdot i' = 0$. Therefore, this algebra is not a real C*-algebra.

If for $x = a+ib \in A_c$ $(a, b \in A)$ we put $\alpha(x) = \alpha(a+ib) = a^*+ib^*$, then, it is straightforward to check that α: $A_c \to A_c$ is a *-anti-automorphism of period 2 on A_c, i.e., α is linear, keeps the involution, $\alpha(xy) = \alpha(y)\alpha(x)$ for $x, y \in A_c$ and $\alpha^2 = id$, i.e., $\alpha(\alpha(x)) = x$ for all $x \in A_c$, and moreover:

$$A = \{x \in A_c : \alpha(x) = x^*\}$$

To motivate the next definitions, suppose A is a *-ring with unity, and let $w \in A$ be a partial isometry. If $e = w^* w$, it results from $w = w w^* w$ that $wy = 0$ iff $ey = 0$ iff $(1-e) y = y$ iff $y \in (1-e)A$, thus, the elements that right-annihilate w form a principal right ideal generated by a projection. The idea of a Rickart *-ring (defined below) is that such a projection exists for every element w (not just the partial isometries).

Definition 2.2. If A is a ring and S is a nonempty subset of A, we write:

$$R(S) = \{x \in A : sx = 0, \ \forall s \in S\}$$

and call $R(S)$ the *right-annihilator* of S. Similarly,

$$L(S) = \{x \in A : xs = 0, \ \forall s \in S\}$$

Denotes the *left annihilator* of S.

Definition 2.3. A *Rickart *-ring* is a *-ring A such that, for each $x \in A$, $R(\{x\}) = gA$ with g a projection (note that such a projection is unique). It follows that $L(\{x\}) = (R(\{x^*\}))^* = (hA)^* = Ah$ for a suitable projection h. A (complex) C*-algebra that is a Rickart *-ring will be called a *Rickart C*-algebra*. A real C*-algebra that is a Rickart *-ring will be called a *Rickart real C*-algebra*.

THE CONNECTION OF A RICKART REAL C*-ALGEBRA WITH ITS ENVELOPING RICKART (COMPLEX) C*-ALGEBRA

This section considers the connection of a Rickart real C*-algebra with its enveloping Rickart (complex) C*-algebra. There is an example of a Rickart real C*-algebra for which the enveloping C*-algebra (i.e., its complexification) is not a Rickart (complex) C*-algebra.

Example 3.1. [2, Exercise 14A]. Let B be the set of all sequences $x = \{\lambda_n\}$ of complex numbers such as $|\lambda_n|$ is bounded and $\mathrm{Im}(\lambda_n) \to 0$ (i.e., bounded sequences which are "real at infinity"). With coordinate-wise operation and $x^* = \{\overline{\lambda_n}\}$, B is a real *-algebra. Setting $\|x\| = \sup\{|\lambda_n|\}$, we obtain that B is a real C*-algebra, since $\|x^*x\| = \|x\|^2$ and $1 + x^*x$ is invertible, for any $x \in B$, where $1 = \{1\}$ is the identity in B. Projections in B are sequences of 0's and 1's. Using [7, Proposition 4.2.2], similarly to [2, Example 5], we can show that B is a Rickart real C*-algebra.

By applying the scheme of proof of [7, Proposition 4.2.3] we obtain:

Theorem 3.1. *The complex C*-algebra B+iB is not a Rickart C*-algebra.*

Now, let us consider the converse problem: *If A is a real C*-algebra and A+iA is a Rickart C*-algebra is A necessarily a Rickart real C*-algebra?*

The following result gives a positive answer to this problem, which is the main result of this section.

Theorem 3.2. *Let A be a real C*-algebra and let $A_c = A + iA$ be its complexification. Suppose that A_c is a Rickart C*-algebra. Then, A is a Rickart real C*-algebra.*

Proof. If $a \in A$, then, for set $\{a\}$ right-annihilator (with respect to A_c), we have:

$$R_c(\{a\}) = \{x \in A_c : ax = 0\}$$

and

$$x \in R_c(\{a\}) \Leftrightarrow ax = 0 \Leftrightarrow 0 = \alpha(ax)^* = \alpha(a)^* \alpha(x)^* = a(\alpha x)^*,$$

because $\alpha(a)^* = a \in A$. This means that $x \in R_c(\{a\})$ if and only if $\alpha(x)^* \in R_c(\{a\})$.

Now, suppose that A_c is a Rickart C*-algebra, then, $R_c(\{a\}) = gA_c$ for a suitable projection $g \in A_c$. Since $g \in R_c(\{a\})$ from above it follows that $\alpha(g^*) = \alpha(g) \in R_c(\{a\})$. Hence, for $h = \alpha(g)$, we have:

$$h^2 = \alpha(g)\alpha(g) = \alpha(g^2) = \alpha(g) = h,\ h^* = \alpha(g)^* = \alpha(g^*) = \alpha(g) = h$$

Therefore, $h = \alpha(g)$ is a projection and $\alpha(g) \in gA_c$, i.e., $\alpha(g) = g\alpha(g)$ ($h = gh$). Thus:

$$g = \alpha(\alpha(g)) = \alpha(h) = \alpha(gh) = \alpha(h)\alpha(g) = g\alpha(g) = \alpha(g).$$

This means that $g \in A$. But then:

$$R(\{a\}) = R_c(\{a\}) \cap A = gA_c \cap A = gA,$$

i.e., A is a Rickart real C*-algebra. This completes the proof of the theorem.

MAIN RESULTS

In this section, we will show that there is a Rickart real C*-algebra whose projection lattice is not complete. We will consider the connection between Rickart's real C*-algebra and real AW*-algebra.

Proposition 4.1. [2, Proposition 3, paragraph 3]. *Let A be a Rickart *-ring and* $x \in A$. *There exists a A unique projection exists such as (1)* $xe = x$, *and (2)* $xy = 0$ *iff* $ey = 0$. *Similarly, there exists a unique projection exists such as (3)* $fx = x$, *and (4)* $yx = 0$ *iff* $yf = 0$. *Explicitly* $R(\{x\}) = (1\text{-}e)A$ *and* $L(\{x\}) = A(1\text{-}f)$. *The projections e and f are minimal in the properties (1) and (3), respectively*. We write $e = RP(x)$, $f = LP(x)$, called the *right projection* and the *left projection* of x.

It is known that any C*-algebra can be isomorphically embedded into some $B(H)$. Namely, there is an isomorphism of a C*-algebra onto a uniformly closed C*-subalgebra $B(H)$, for some complex Hilbert space H. On the other hand, if $A \subset B(H)$ be a real C*-algebra, then, it is also known that (see Proposition 5.1.2 [6]), there is a real Hilbert space H with:

$H_r + iH_r = H, \quad A \subset B(H_r) \subset B(H_r) + iB(H_r) = B(H).$

Proposition 4.2. *The real C*-algebra* $B(H_r)$ *is a Rickart real C*-algebra. Explicitly, if* $x \in B(H_r)$ *then* $LP(x)$ *is the projection on the closure of the range of x, and* $1\text{-}RP(x)$ *is the projection on the null space of x* (*i.e.,* $1\text{-}RP(x) \in x^* (H_r)^{\perp}$).

Proof. Let $x \in B(H_r)$. Let $f\colon H_r \to \overline{x(H_r)}$ be a projection. It is obvious that if $sx = 0$, then, $s = 0$ on $x(H_r) \Leftrightarrow s = 0$ on $\overline{x(H_r)}$. Then, $sf = 0$, and hence, we have $s(1\text{-}f) = s$ and $s \in B(H_r)(1\text{-}f)$. Thus, we obtain $L(\{x\}) = B(H_r)(1\text{-}f)$. This shows that $B(H_r)$ is a Rickart *-ring. It follows that $RP(x) = LP(x^*)$ is the projection on $\overline{x^*(H_r)}$, therefore, $1\text{-}RP(x)$ is the projection on $\overline{x^*(H_r)}^{\perp} = x^*(H_r)^{\perp}$, which is the null space of x. This completes the proof of the proposition. ■

If A is a Rickart *-ring and B is a *-subring of A, then, B need not be a Rickart *-ring: An obvious example is when B has no unity element (a Rickart *-ring always has a unity element), but adjoining a unity element may not be a remedy (we will show it below). There is, nevertheless, a useful positive result.

Proposition 4.3. [2, Proposition 8, paragraph 3]. *Let A be a Rickart *-ring and let B be a *-subring such that (1) B has a unity element, and (2)* $x \in B$ *implies* $RP(x) \in B$. *Then, B is also a Rickart *-ring.*

Theorem 4.1. *There exists a Rickart real C*-algebra whose projection lattice is not complete.*

Proof. Let H_r be an inseparable real Hilbert space, and let $A \subset B(H_r)$ be the subset of all operators with separable range, i.e., $a(H_r)$, a separable, for $a \in A$. Let A_1 be its unification:

$$A_1 = \{a + \lambda 1 : a \in A, \lambda \in \mathbb{R}$$

Where 1 is the identity operator on H_r. It is directly proved that A_1 is a real C*-algebra. We show it is a Rickart *-ring with an incomplete projection lattice.

To prove that A_1 is a Rickart *-ring, by Proposition 4.3, it will suffice to show that $x \in A_1$ implies $RP(x) \in A_1$. Let $x = a + \lambda 1 \in A_1$ (where $a \in A$, $\lambda \in \mathbb{R}$). If $\lambda = 0$, then, $RP(x) = RP(a) = LP(a^*) \in A$ because the closure of a separable linear subspace is separable. Let $\lambda \neq 0$. By Proposition 4.2, the algebra $B(H_r)$ is a Rickart *-ring and $g = 1 - RP(x)$ is the projection on the null space of x. Then, $xg = 0$, i.e., $(a + \lambda 1)g = 0$. Hence, $g = a(-\lambda^{-1})g \in A$ because A is an ideal in $B(H_r)$, therefore, we obtain $RP(x) = 1 - g \in A_1$. By Proposition 4.3, the algebra A_1 is a Rickart *-ring.

Finally, the projection lattice of A_1 is incomplete. For example, if $e \in B(H_r)$ is any projection such that both e and $1\text{-}e$ have an inseparable range, then, the separable sub projections of e can have no supremum in A_1. This completes the proof of the theorem.■

Definition 4.1. A *Bear *-ring* is a *-ring A such that for every nonempty subset $S \subset A$, $R(S) = gA$ for a suitable projection g. It follows that $L(S) = R(S^*)^* = (hA)^* = Ah$ for a suitable projection h.

The relation between Rickart *-rings and Baer *-rings is the relation between lattices and complete lattices:

Proposition 4.4. [2, Proposition 1, paragraph 4]. *The following conditions on a *-ring A are equivalent:*

(a) *A is a Baer *-ring;*

(b) *A is a Rickart *-ring whose projections form a complete lattice;*

(c) *A is a Rickart *-ring in which every orthogonal family of projections has a supremum.*

Definition 4.2. A (complex) C*-algebra that is a Bear *-ring will be called an *AW*-algebra*. A real C*-algebra that is a Bear *-ring will be called a *real AW*-algebra*.

It is known that $B(H)$ and $B(H_r)$ are complex and real AW*-algebras, respectively (see [2, Proposition 1, paragraph 4] and [7, Corollary 4.6.12]). In particular, any complex and real W*-algebra is complex and real AW*-algebra, respectively. Recall that [8] *W*-algebra* is a weakly closed complex *-algebra of operators on a Hilbert space H containing the identity operator 1 (W*-algebras are also called *von Neumann algebras*).

By Theorem 4.1 and Proposition 4.4, we obtain:

Corollary 4.1. *A real AW*-algebra is a Rickart real C*-algebra, but the converse is not true, i.e., a Rickart real C*-algebra does not need to be a real AW*-algebra.*

Corollary 4.2. *The algebra B from the example above is a Rickart real C*-algebra but by Theorem 3.1, it is not a real W*-algebra.*

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

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