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GEOMETRIC PROPERTIES OF MULTIVALENT FUNCTIONS ASSOCIATED WITH PARABOLIC REGIONS

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ABSTRACT

The main purpose of this article is to derive the connections between the parabolic starlike and parabolic uniformly convex functions by applying an integral operator on multivalent functions. In addition, a parabolic region in the half-plane is introduced to study the family of parabolic multivalent convex functions of order α and type β.

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INTRODUCTION

The subclass of multivalent functions has attraction many researchers in the area of geometric function theory nowadays. This area of research has motivated the researchers recently since of its numerous applications in applied sciences, e.g., optimal control problems, ordinary fractional calculus, q-transform analysis, q-difference and q-integral equations, q-derivative, q-Chebyshev polynomials, etc., see [1] and [4]. Since then this area has grown into many directions and many of its subfamilies were developed recently, such as by using the concept of the q-calculus in association with the Janowski functions [3], etc. Before stating and proving our main results, we give a brief discussion on the basics of this area which will be beneficial in understanding the work to follow.

Let A_p denote the class of functions in the form of

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1}$$

which is analytic and multivalent (or p-valent) in the open unit disk $D = \{z \in C{:}|z|1\}$. Also $f(z) \in A_p$ is said to be in the class PS of parabolic starlike functions in D if

$$\left|\frac{zf'}{f} - p\right| < Re\left\{\frac{zf'}{f}\right\}, \quad (z \in D), \tag{2}$$

and $f(z) \in A_p$ is said to be in the class PUC of parabolic uniformly convex functions in D if

$$\left|\frac{z f''}{f'} + 1 - p\right| < Re\left\{1 + \frac{z f''}{f'}\right\}, \quad (z \in D), \tag{3}$$

see [5] and [8]. In other words, for every circular arc contained in *D*, with a centre also in *D*, the image of the curve under *f* is a convex arc. The various properties of parabolic starlike and parabolic uniformly convex functions were investigated by many authors. For example, see [2], [6], [7] and [9].

For $c > -p$ (*c,p are real numbers*), we consider the integral operator:

$$F_c(f) = (c + p)\int_0^z t^{c-1} f(t)\, dt, \tag{4}$$

where $f(z)$ is given in (1). For convenience, we write $(F_c(f))'z^{-c} = F$, see [10]. Motivated from the discussion above, we utilize the concepts of p-valent functions associated with parabolic regions and investigate some of their geometric properties.

MAIN RESULT

In this section, we consider the families *PS*, *PUC* and relations between them. Such classes were studied in [10]. See also [11] and [12].

Theorem 1. Let $f(z) \in A_p$, then F is in *PUC* if and only if $f(z) \in PS$.

Proof. From (4), we get:

$$F' = \frac{c+p}{z} f(z), \tag{5}$$

and

$$F'' = (c+p)\left(\frac{zf'(z) - f(z)}{z^2}\right) \tag{6}$$

Then by (3), we have $F \in PUC$ if and only if

$$\left|\frac{zF''}{F'} + 1 - p\right| < Re\left\{1 + \frac{zF''}{F'}\right\}, \tag{7}$$

or equivalently, by putting (5) and (6) in the above inequality, we get:

$$\left|\frac{(c+p)\left(\frac{z f'(z) - f(z)}{z}\right)}{(c+p)\frac{f(z)}{z}} + 1 - p\right| < Re\left\{1 + \frac{(c+p)\left(\frac{z f'(z) - f(z)}{z}\right)}{(c+p)\frac{f(z)}{z}}\right\}. \tag{8}$$

Thus

$$\left|\frac{z f'(z)}{f(z)} - p\right| < Re\left\{\frac{z f'(z)}{f(z)}\right\}, \tag{9}$$

then by (2), $f(z)$ is a parabolic starlike function and so $f(z) \in PS$.

A function $f(z) \in A_p$ is said to be a member of $PMC(\alpha,\beta)$ of parabolic multivalent convex functions of order α and type β, if the following condition holds:

$$\left|\frac{zF''}{F'} + 1 - p(\alpha+\beta)\right| < p(\beta-\alpha) + Re\left\{1 + \frac{zF''}{F'}\right\}$$

where $0 \leq \alpha < 1, 0 < \beta < +\infty$, $\alpha \leq \beta$ and F is given in (4).

Theorem 2. A function $f \in PMC(\alpha,\beta)$ if and only if for every $z \in D$, the values of $\frac{zF''}{F'} + 1$ lie in the interior of the parabolic region.

Proof. By (7), if we put the values of $\frac{zF''}{F'} + 1$ equal to W, we have:

$$|W - p(\alpha+\beta)| < p(\beta-\alpha) + Re\{w\},$$

or

$$|Re\{w\} - p(\alpha+\beta)|^2 + (Im\{w\})^2 < (p(\beta-\alpha) + Re\{w\})^2,$$

or

$$(Im\{w\})^2 < \big(2p(\alpha+\beta) + 2p(\beta+\alpha)\big)(Re\{w\}) - 4p^2\alpha\beta.$$

Hence

$$(Im\{w\})^2 < 4p\beta(Re\{w\} - p\alpha),$$

and that is the interior of the parabolic region in the half-plane (right side) with the vertex at $(p\alpha,0)$ and $4p\beta$ is the length of the latus rectum. The relations are reversible, so the proof is complete.

Theorem 3. Let $f(z) \in A_p$, and F are defined by (4). Then f is p-valently starlike of order λ if and only if F is p-valently convex of order λ.

Proof. Let F be p-valently convex of order λ, then:

$$Re\left\{1 + \frac{zF''}{F'}\right\} > \lambda.$$

But by (5) and (6), we get:

$$Re\left\{1 + \frac{zF''}{F'}\right\} = Re\left\{1 + \frac{z f' - f}{f}\right\} = Re\left\{\frac{z f'}{f}\right\} > \lambda,$$

and so $f(z)$ is p-valently starlike. The relations are reversible and so conclude the required result.

In the last theorem, we show that the family $PMC(\alpha,\beta)$ is closed under the product of functions with real powers.

Theorem 4. *Let* $f_j(z) \in PMC(\alpha_j,\beta_j)$, with $0 \le a_j < 1$, $\sum_{j=1}^{m} a_j < 1$, $0 < \beta_j < \infty$ and $j = 1,2,...,m$. Then $g(z) = \Pi_{j=1}^{m} (f_j)^{dj}$ is in $PMC(\alpha,\beta)$, where $\alpha = \sum_{j=1}^{m} d_j \alpha_j$ and $\beta = \sum_{j=1}^{m} d_j \beta_j$.

Proof. Since $f_j \in PMC(\alpha_j,\beta_j)$, $j = 1,2,...,m$, then by (7), we have:

$$\left|\frac{zF''}{F'} + 1 - p(\alpha_j + \beta_j)\right| < p(\beta_j - \alpha_j) + Re\left\{1 + \frac{zF''}{F'}\right\}. \tag{8}$$

Now, we must show that:

$$\left|\frac{zG''}{G'} + 1 - p(\alpha+\beta)\right| < p(\beta-\alpha) + Re\left\{1 + \frac{zG''}{G'}\right\},$$

where

$$G = G_c(g) = \frac{c + p}{z^c} \int_0^z \quad t^{c-1}\, g(t)\ dt,$$

$$\text{and } \ g(z) = \prod_{j=1}^{m} \quad (f_j)^{d_j}.$$

By a direct calculation, we obtain:

$$\begin{aligned} \left| \frac{zG''}{G'} + 1 - p(\alpha + \beta) \right| &= \left| \frac{zg'}{g} - p(\alpha + \beta) \right| \\ &= \left| \sum_{j=1}^{m} \quad d_j \left(\frac{zf'_j}{f_j} - p(\alpha_j, \beta_j) \right) \right| \\ &\leq \sum_{j=1}^{m} \quad \left(d_j \left| \frac{zf'_j}{f_j} - p(\alpha_j + \beta_j) \right| \right). \end{aligned}$$

With a simple calculation on (8), we get:

$$\left| \frac{z\,f'}{f} - p(\alpha_j, \beta_j) \right| < Re\left\{ \frac{z\,f'}{f} \right\} + p(\beta_j, \alpha_j) \qquad (j = 1,2,\dots,m),$$

and so

$$\begin{aligned} \left| \frac{zG''}{G'} + 1 - p(\alpha + \beta) \right| &< \sum_{j=1}^{m} \quad \left(d_j \left(Re\left\{ \frac{zf_j'}{f_j} \right\} + p(\alpha_j + \beta_j) \right) \right) \\ &= Re\left\{ \frac{zg'}{g} \right\} + p(\beta - \alpha). \end{aligned}$$

Hence, $g(z) \in PMC(\alpha,\beta)$ and this gives the required result.

CONCLUSION

The aim of this paper is to derive the connections between the families of parabolic starlike and parabolic uniformly convex functions by applying the integral operator on multivalent functions, which is the recent attraction for many researchers nowadays. A parabolic region in the half-plane is introduced to study the property for the family of parabolic multivalent convex functions of order α and type β, i.e., $PMC(\alpha, \beta)$. Finally, we obtained that, the family $PMC(\alpha, \beta)$ is closed under the product of functions with real powers.

CONFLICTS OF INTEREST

The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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