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A REMARK ON THE EDGE IRREGULARITY STRENGTH OF CORONA PRODUCT OF TWO PATHS

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ABSTRACT

With respect to a simple graph G, a vertex labeling ϕ*:* $V(G) > \{1,2,...,$ *k)* is known as k-labeling. The weight corresponding to an edge xy in G, expressed as $w_\phi\,(xy)$, represents the labels sum of end vertices x and y, given by $w_\phi\,(xy) = \phi(x) + \phi(y)$ A vertex k-labeling is expressed as an edge irregular k-labeling with respect to graph G provided that for every two distinct edges e and f, there exists $w_\phi(e) \neq w_\phi(f)$ Here, the minimum k where the graph G possesses an edge irregular k-labeling is known as the edge irregularity strength with respect to G, expressed as (G). Here, we examine the edge irregularity strength's exact value of corona product with respect to two paths P_n and P_m , in which $n \geq 2$ and $m = 3, 4, 5$.

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INTRODUCTION

Suppose G is a connected, simple as well as an undirected graph having vertex set $V(G)$ with edge set $E(G)$. Here, the mapping of a set of graph elements onto a set of numbers (particularly positive integers) is referred to as *labeling*. Moreover, the *labelings* are expressed as *edge labelings* or *vertex labelings*, depending on whether the domain refers to the edge or the vertex set, accordingly. The labeling is denoted as a *total labeling* provided that the domain is given by $V(G) \cup E(G)$. Therefore, for any edge k-labeling $\delta{:}E(G)\rightarrow\{1,2,..., k\}$, the corresponding weight of a vertex $x \in (G)$ is given by

$$w_\delta(x) = \sum\delta(xy),$$

in which the sum is taken with respect to all vertices adjacent to x.

Chartrand *et al.* [9] established the edge k-labeling δ with respect to a graph G provided that $w_\delta(x) = \sum\delta(xy)$ for every vertices $x,y \in V(G)$ having $x \neq y$. These labelings were expressed as *irregular assignments* with minimum k while G possessing an irregular assignment employing labels at most is known as the *irregularity strength* $s(G)$ with respect to a graph G.

Moreover, Baca *et al.* [6] expressed a *vertex irregular total k-labeling* with respect to a graph G to represent a total labeling of G given by $\psi{:}V(G) \cup E(G)\rightarrow\{1,2,..., k\}$, , in which the *total vertex-weights* may be expressed as

$$wt(x) = \psi(x) + \sum_{xy\in E(G)} \psi(xy),$$

and are distinct for all corresponding vertices. In other words, $wt(x) \neq wt(y)$ for all distinct vertices $x, y \in V(G)$. Here, the minimum k for G to have a vertex irregular total k-labeling represents the *total vertex irregularity strength* with respect to G, expressed as $tvs(G)$. The authors also expressed the total labeling ψ: $V(G)\cup E(G)\rightarrow\{1,2,...,k\}$ to represent an edge irregular total k-labeling with respect to the graph G provided that for every two distinct edges xy and $x'y'$ of G, we have $wt(xy) = \psi(x)+ \psi(xy)+ \psi(y) \neq wt(x'y') = \psi(x') + \psi(x'y') + \psi(y')$. Moreover, the *total edge irregularity strength*, $tes(G)$, is expressed as the minimum k where G possesses an edge irregular total k-labeling. [8] provides the latest and most comprehensive review of graph labelings.

A vertex k-labeling ϕ: $V(G)\rightarrow\{1,2,...,k\}$ is expressed as an *edge irregular k-labeling* with respect to the graph G provided that for every two distinct edges e and f, there exists $w_\phi(e) \neq w_\phi(f)$, Here, the edge's weight $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x)+ \phi(y)$. Moreover, the minimum k where the graph G possesses an edge irregular k-labeling is denoted as the *edge irregularity strength* with respect to G, expressed by $es(G)$ [1].

The authors of [1] calculated the edge irregularity strength's exact values (es) for numerous graph families, including stars, paths, double stars, as well as the Cartesian product with respect to two pathways. Moreover, Mushayt [5] examined the edge irregularity strength with respect to Cartesian product of star, cycle corresponding to path P_2 as well as strong product of path P_n with P_2. Furthermore, Tarawneh *et al.* [10-12] examined the exact value with respect to edge irregularity strength of corona product for graphs having paths, cycles as well as cycle having isolated vertices. In addition, Ahmad [2] investigated the edge irregularity strength's the exact value of corona graph $C_n \odot mk_1$ (or called the sun graph S_n). Also, Ahmad *et al.* [3] examined the edge irregularity strength's exact value with respect to various classes of Toeplitz graphs. Subsequently, Tarawneh *et al.* [13] examined the edge irregularity strength with respect to disjoint union for star graph including its subdivision. Meanwhile, Imran *et al.* [9] examined the edge irregularity strength's exact value with respect to caterpillars, (n,t)-kite graphs, n-star graphs, cycle chains as well as friendship graphs. Moreover, Ahmad *et al.* [4] examined the edge irregularity strength with respect to several chain graphs including the joint concerning two graphs.

The theorem stated below establishes the lower bound with respect to the edge irregularity strength for a graph G.

Theorem 1. *[1] Suppose $G = (V,E)$ denote a simple graph having maximum degree $\Delta = \Delta(G)$. We then have*

$$es(G) \geq \left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}.$$

In [10], the authors investigated the exact value with respect to the edge irregularity strength of corona product for path P_n that is P_2, P_n having S_m, in which $\{n \geq 2, m \geq 3\}$. This paper discovers the exact value with respect to edge irregularity strength of corona product for path P_n with P_3, P_n with P_4 as well as P_n with P_5, in which $n \geq 2$.

MAIN RESULTS

The corona product with respect to two graphs G as well as H, expressed by $G \odot H$, denotes the graph yielded by employing one copy of G (with n vertices) and n copies $H_1, H_2,..., H_n$ of H. Then, the G's i-th vertex is joined to each vertex in H_i.

The corona product $P_n \odot P_m$ denotes a graph having the vertex set $V(P_n \odot P_m) = \{x_i, y_i^j: 1 \leq i \leq j \leq m)$ as well as edge set $E(P_n \odot P_m) = \{x_i x_{i+1}, 1 \leq i \leq n-1\} \cup \{x_i y_i^j: 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y_i^j y_i^{j+1}: 1 \leq i \leq n, 1 \leq j \leq m-1\}$.

Below, we consider the exact value with respect to edge irregularity strength for $P_n \odot P_m$ for $n \geq 2$ and $m = 3,4,5$.

Lemma 2. *For any integer $n \geq 2$, $es(P_n \odot P_m)$=3n+1.*

Proof. Assume that $P_n \odot P_3$ is a graph having a vertex set denoted by $V(P_n \odot P_3) = \{x_i, y_i^{j+1}: 1 \leq i \leq n, 1 \leq j \leq 3$ as well as the edge set $E(P_n \odot P_3) = \{x_i x_{i+1}: 1 \leq i \leq n-1\}, \cup \{x_i y_i^j: 1 \leq i \leq n, 1 \leq j \leq 3\} \cup \{y_i^j y_i^{j+1}: 1 \leq i \leq n, 1 \leq j \leq 2\}$.

According to Theorem 1, we have that $es(P_n \odot P_3) \geq 3_n$. Since every edge $E(P_n \odot P_3) \setminus \{x_i, x_{i+1}\}$ for $1 \leq i \leq n-1$ denote a portion of complete graph K_3, with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Thus, the smallest edge weight 2 as well as the largest edge weight $6n$ will be of edges x_i, x_{i+1}. For this there will be two pair of adjacent vertices, for instance one pair of adjacent vertices assigned label 1, a second pair of adjacent vertices assigned label $3n$, then there will be two distinct edges having the same weight. Therefore $es(P_n \odot P_3) \geq 3n+1$. To show that $es(P_n \odot P_3) \leq 3n+1$, we now express a vertex labeling $\phi_1 (P_n \odot P_3) \rightarrow \{1,2,...,3n+1\}$ as follows:

$$\phi_1(x_i) = 2\left(i + \lfloor\frac{i-1}{2}\rfloor\right) + 1 \quad if\ 1 \leq i \leq n$$

and

$$\phi_1\left(y_i^j\right) = \{3(i-1) + j + \lfloor\frac{j-1}{2}\rfloor, \quad if\ i\ is\ odd, 1 \leq j \leq 3\ 3i + j + \lfloor\frac{j}{2}\rfloor - 3, \quad if\ i\ is\ even, 1 \leq j \leq 3$$

since

$$w_{\phi_1}(x_i x_{i+1}) = \phi_1(x_i) + \phi_1(x_{i+1}) = 6i + 2 \quad for\ 1 \leq i \leq n-1,$$

$$w_{\phi_1}\left(x_i y_i^j\right) = \{5i + 2\lfloor\frac{i-1}{2}\rfloor + \lfloor\frac{j-1}{2}\rfloor + j - 2, \quad if\ i\ is\ odd, 1 \leq j \leq 3\ 5i + 2\lfloor\frac{i-1}{2}\rfloor + \lfloor\frac{j}{2}\rfloor + j - 2, \quad if\ i\ is\ even, 1 \leq j \leq 3$$

and

$$w_{\phi_1}\left(y_i^j y_i^{j+1}\right) = \{6(i-1) + 3j, \quad if\ i\ is\ odd, 1 \leq j \leq 2\ 6i + 3j - 5, \quad if\ i\ is\ even, 1 \leq j \leq 2$$

Thus, the edge weights are distinct with respect to all pairs of different edges. Hence, the vertex labeling ϕ_1 denotes an optimal edge irregular $(3n+1)$-labeling, which then fullfils the proof.

Example 1. In Figure 1, we present the *es* labeling for graph $P_5 \odot P_3$ with vertex labels and edge weight for the case n=5 .

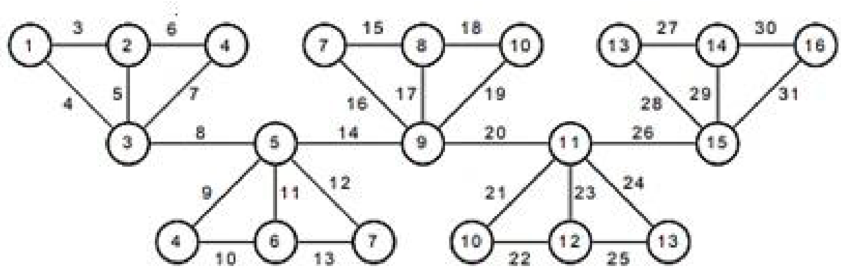

Figure 1: Graph $P_5 \odot P_3$ with *es* labelling

Lemma 3. *For any integer $n \geq 2$, es $(P_n \Theta P_4)$ =* 4n +1.

Proof. Let $P_n \Theta P_4$ express a graph having vertex set given by $V(P_n \Theta P_4) = \{x_i y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq 4\}$ and the edge set $E(P_n \Theta P_4) = \{x_i x_{i+1}, 1 \leq i \leq n-1\} \cup \{x_i y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq 4\} \cup \{y_i^j y_i^{j+1} : 1 \leq i \leq n,\ 1 \leq j \leq 3\}$.

According to Theorem 1, we have that $es(P_n \Theta P_4) \geq 4n$. Since every edge $E(P_n \Theta P_4) \setminus \{x_i x_{i+1}\}$ for $1 \leq i \leq n-1$ denote a portion of complete graph K_3, with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Therefore, the smallest edge weight 2 and the largest edge weight $8n$ will be of edges $x_i x_{i+}$. For this there will be two pairs of adjacent vertices such that one pair of adjacent vertices assign label 1, a second pair of adjacent vertices assign label 4_n, then there will be two distinct edges having the same weight. Therefore $es(P_n \Theta P_4) \geq 4_n+1$. To show that $es(P_n \Theta P_4) \leq 4_n+1$, we define a vertex labeling ϕ_2 $(P_n \Theta P_4) \rightarrow \{1,2,...,4n+1\}$ as given below:

$$\phi_2(x_i) = 4i-1, \text{ if } 1 \leq i \leq n$$

and

$$\phi_2(y_i^j) = \{4(i-1)+j, \quad if\ 1 \leq i \leq n \ \ and\ \ 1 \leq j \leq 2\ 4i+j-3, \quad if\ 1 \leq i \leq n \ \ and\ \ 3 \leq j \leq 4$$

since

$$w_{\phi_2}(x_i x_{i+1}) = \phi_2(x_i) + \phi_2(x_{i+1}) = 2(4i+1) \quad for\ 1 \leq i \leq n-1,$$

$$w_{\phi_2}(x_i y_i^j) = \{8i+j-5, \quad if\ 1 \leq i \leq n \ \ and\ \ 1 \leq j \leq 2\ 8i+j-4, \quad if\ 1 \leq i \leq n \ \ and\ \ 3 \leq j \leq 4$$

and

$$w_{\phi_2}(y_i^j y_i^{j+1}) = \{8i-5, \quad if\ 1 \leq i \leq n \ \ and\ \ j=1\ 8i-2, \quad if\ 1 \leq i \leq n \ \ and\ \ j=2\ 8i+1, \quad if\ 1 \leq i \leq n \ \ and\ \ j=3.$$

Thus, the edge weights are distinct with respect to all pairs of different edges. Therefore, the vertex labeling ϕ_2 denotes an optimal edge irregular (4_n+1)-labeling, which then completes the proof.

Example 2. In Figure 2, we present the *es* labelling for graph $(P_4 \Theta P_4)$ with vertex labels for the case n=4.

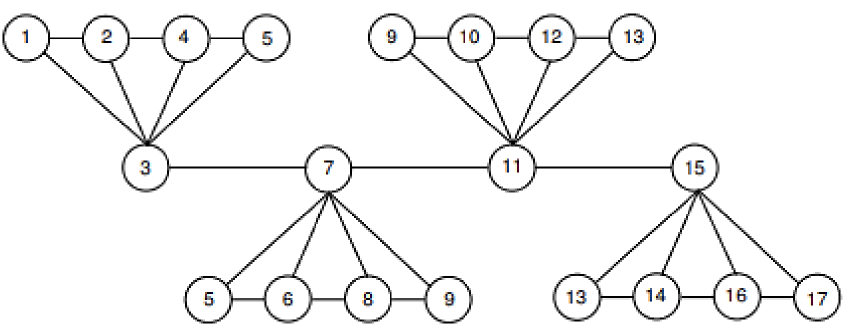

Figure 2: Graph $(P_4 \Theta P_4)$ with *es* labelling

Lemma 4. *For any integer $n \geq 2$,$es(P_n \Theta P_5)$.= $5n+1$.*

Proof. Let $P_n \Theta P_5$ express a graph having the vertex set $V(P_n \Theta P_5) = \{x_i, y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq 5\}$ and the edge set $E(P_n \Theta P_5) = \{x_i, x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_i^j, y_i^{j+1} : 1 \leq i \leq n, 1 \leq j \leq 4\}$. According to Theorem 1, we have that $es(P_n \Theta P_5) \geq 5$. Since every edge $E(P_n \Theta P_5) \backslash \{x_i, x_{i+1}\}$ for $1 \leq i \leq n-1$ represent a portion of complete graph k_3, then, with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Therefore, the smallest edge weight 2 and the largest edge weight $10n$ will be of edges x_i, x_{i+1}. For this there will be two pairs of adjacent vertices such that one pair of adjacent vertices assign label 1, second pair of adjacent vertices assign label $5n$, then there will be two distinct edges having the same weight. Therefore $es(P_n \Theta P_5) \geq 5n+1$. To show that $es(P_n \Theta P_5) \leq 5n+1$, we define a vertex labeling $\phi_3(P_n \Theta P_5) \rightarrow \{1,2,..., 5n+1\}$ as given below:

$$\phi_3(x_i) = 2\left(2i + \lfloor\frac{i-1}{2}\rfloor\right), \quad if\ 1 \leq i \leq n$$

and

$$\phi_3\left(y_i^j\right) = \{5i - j - 2, \quad for\ odd\ i, \quad j = 1,2\ 5i - 2, \quad for\ odd\ i, \quad j = 3\ 5i + j - 4, \quad for\ odd\ i, \quad j = 4,5\ 5i + j - 5, \quad for\ even\ i, \quad j = 1,2\ 5i - 1, \quad for\ even\ i, \quad j = 3\ 5i + 5 - j, \quad for\ even\ i, \quad j = 4,5$$

since

$$w_{\phi_3}(x_i x_{i+1}) = \phi_3(x_i) + \phi_3(x_{i+1}) = 10i + 2 \quad for\ 1 \leq i \leq n-1,$$

$$w_{\phi_3}\left(x_i y_i^j\right) = \{9i - j + 2\lfloor\frac{i-1}{2}\rfloor - 2, \quad for\ odd\ i, \quad j = 1,2\ 9i + 2\lfloor\frac{i-1}{2}\rfloor - 2, \quad for\ odd\ i, \quad j = 3\ 9i + j + 2\lfloor\frac{i-1}{2}\rfloor - 4, \quad for\ odd\ i, \quad j = 4,5\ 9i + j + 2\lfloor\frac{i-1}{2}\rfloor - 5, \quad for\ even\ i, \quad j = 1,2\ 9i + 2\lfloor\frac{i-1}{2}\rfloor - 1, \quad for\ even\ i, \quad j = 3\ 9i + 2\lfloor\frac{i-1}{2}\rfloor - j + 5, \quad for\ even\ i, \quad j = 4,5$$

and

$$w_{\phi_3}\left(y_i^j y_i^{j+1}\right) = \{10i - 7, \quad for\ odd\ i, \quad j = 1\ 10i - 6, \quad for\ odd\ i, \quad j = 2\ 10i - 2, \quad for\ odd\ i, \quad j = 3\ 10i + 1, \quad for\ odd\ i, \quad j = 4\ 10i - 7, \quad for\ even\ i, \quad j = 1\ 10i - 4, \quad for\ even\ i, \quad j = 2\ 10i, \quad for\ even\ i, \quad j = 3\ 10i + 1, \quad for\ even\ i, \quad j = 4$$

Thus, the edge weights are distinct with respect to all pairs of different edges. Hence, the vertex labeling ϕ_3 refers to an optimal edge irregular $(5n+1)$-labeling, which fullfils the proof.

Example 2. In Figure 3, we present the *es* labelling for graph $(P_4 \Theta P_5)$ with vertex labels for the case $n=4$.

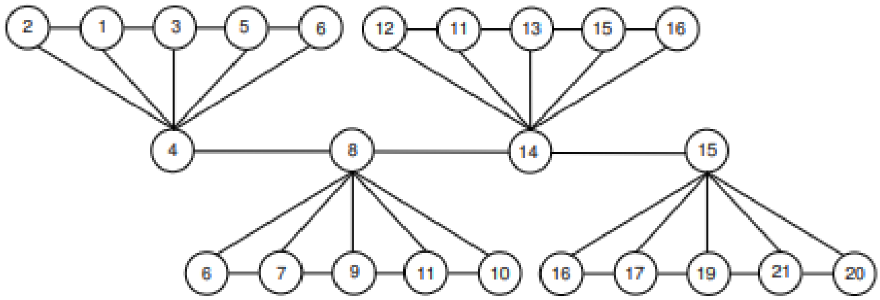

Figure 3: Graph $(P_4 \Theta P_5)$ with *es* labelling

The following main result follows immediately from Lemmas 2, 3 and 4.

Theorem 5. *For any real number* $n \geq 2$ *and* m = *3,4,4, es* $(P_n \Theta P_m) = 5n+1$.

CONCLUSION

This study presents the exact values for edge irregularity strength with respect to corona graphs for path P_n with P_3, P_n with P_4 and P_n with P_5. Recently, the case of corona graphs of P_n with P_6 was done by Alrawajfeh *et al.* [14]. For the next research, we propose to work on some generalizations concerning the estimation of upper bound or determination of the exact value with respect to the edge irregularity strength of corona graphs of P_n with P_m for any $n, m \geq 2$.

CONFLICTS OF INTEREST

No conflict of interest is declared by the authors.

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