A REMARK ON THE EDGE IRREGULARITY STRENGTH OF CORONA PRODUCT OF TWO PATHS

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ABSTRACT

With respect to a simple graph $G$, a vertex labeling $\phi: V(G) \to \{1,2,\ldots, k\}$ is known as $k$-labeling. The weight corresponding to an edge $xy$ in $G$, expressed as $w_\phi(xy)$, represents the labels sum of end vertices $x$ and $y$, given by $w_\phi(xy) = \phi(x) + \phi(y)$ A vertex $k$-labeling is expressed as an edge irregular $k$-labeling with respect to graph $G$ provided that for every two distinct edges $e$ and $f$, there exists $w_\phi(e) \neq w_\phi(f)$ Here, the minimum $k$ where the graph $G$ possesses an edge irregular $k$-labeling is known as the edge irregularity strength with respect to $G$, expressed as $(G)$. Here, we examine the edge irregularity strength’s exact value of corona product with respect to two paths $P_n$ and $P_m$, in which $n \geq 2$ and $m = 3, 4, 5$.

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INTRODUCTION

Suppose $G$ is a connected, simple as well as an undirected graph having vertex set $V(G)$ with edge set $E(G)$. Here, the mapping of a set of graph elements onto a set of numbers (particularly positive integers) is referred to as labeling. Moreover, the labelings are expressed as edge labelings or vertex labelings, depending on whether the domain refers to the edge or the vertex set, accordingly. The labeling is denoted as a total labeling provided that the domain is given by $V(G) \cup E(G)$. Therefore, for any edge $k$-labeling $\delta:E(G)\to\{1,2,\ldots, k\}$, the corresponding weight of a vertex $x \in (G)$ is given by

$$w_\phi(x) = \sum \delta(xy),$$

in which the sum is taken with respect to all vertices adjacent to $x$.

Chartrand et al. [9] established the edge $k$-labeling $\delta$ with respect to a graph $G$ provided that $w_\delta(x) = \sum \delta(xy)$ for every vertices $x,y \in V(G)$ having $x \neq y$. These labelings were expressed as irregular assignments with minimum $k$ while $G$ possessing an irregular assignment employing labels at most is known as the irregularity strength $s(G)$ with respect to a graph $G$.

Moreover, Baca et al. [6] expressed a vertex irregular total $k$-labeling with respect to a graph $G$ to represent a total labeling of $G$ given by $\psi:V(G) \cup E(G)\to\{1,2,\ldots, k\}$, in which the total vertex-weights may be expressed as

$$wt(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy),$$
and are distinct for all corresponding vertices. In other words, \( wt(x) \neq wt(y) \) for all distinct vertices \( x, y \in V(G) \). Here, the minimum \( k \) for \( G \) to have a vertex irregular total \( k \)-labeling represents the \textit{total vertex irregularity strength} with respect to \( G \), expressed as \( \text{tvsl}(G) \). The authors also expressed the total labeling \( \phi \) of \( G \), here, the edge’s weight \( e = xy \in E(G) \) is \( w_\phi(e) = \phi(x) + \phi(y) \). Moreover, the \textit{total edge irregularity strength}, \( \text{tes}(G) \), is expressed as the minimum \( k \) where \( G \) possesses an edge irregular total \( k \)-labeling. [8] provides the latest and most comprehensive review of graph labelings.

A vertex \( k \)-labeling \( \phi : V(G) \rightarrow \{1,2,...,k\} \) is expressed as an \textit{edge irregular \( k \)-labeling} with respect to the graph \( G \) provided that for every two distinct edges \( e \) and \( f \), there exists \( w_\phi(e) \neq w_\phi(f) \). Here, the edge’s weight \( e = xy \in E(G) \) is \( w_\phi(xy) = \phi(x) + \phi(y) \). Moreover, the minimum \( k \) where the graph \( G \) possesses an edge irregular \( k \)-labeling is denoted as the \textit{edge irregularity strength} with respect to \( G \), expressed by \( \text{es}(G) \) [1].

The authors of [1] calculated the edge irregularity strength’s exact values (es) for numerous graph families, including stars, paths, double stars, as well as the Cartesian product with respect to two pathways. Moreover, Mushayt [5] examined the edge irregularity strength with respect to Cartesian product of star, cycle corresponding to path \( P^2 \) as well as strong product of path \( P^a \) with \( P^2 \). Furthermore, Tarawneh et al. [10-12] examined the exact value with respect to edge irregularity strength of corona product for graphs having paths, cycles as well as cycle having isolated vertices. In addition, Ahmad [2] investigated the edge irregularity strength’s exact value of corona graph \( C_m \bigcirc m K_t \) (or called the sun graph \( S^t \)). Also, Ahmad et al. [3] examined the edge irregularity strength’s exact value with respect to various classes of Toeplitz graphs. Subsequently, Tarawneh et al. [13] examined the edge irregularity strength with respect to disjoint union for star graph including its subdivision. Meanwhile, Imran et al. [9] examined the edge irregularity strength’s exact value with respect to caterpillars, \((n,t)\)-kite graphs, \( n \)-star graphs, cycle chains as well as friendship graphs. Moreover, Ahmad et al. [4] examined the edge irregularity strength with respect to several chain graphs including the joint concerning two graphs.

The theorem stated below establishes the lower bound with respect to the edge irregularity strength for a graph \( G \).

**Theorem 1.** [1] Suppose \( G = (V,E) \) denote a simple graph having maximum degree \( \Delta = \Delta(G) \). We then have

\[
\text{es}(G) \geq \frac{|E(G)|+1}{2} \Delta(G).
\]

In [10], the authors investigated the exact value with respect to the edge irregularity strength of corona product for path \( P^a \), that is \( P^a \), \( P^a \) having \( S^m \), in which \( \{n \geq 2, m \geq 3\} \). This paper discovers the exact value with respect to edge irregularity strength of corona product for path \( P^a \) with \( P^b \), \( P^c \) with \( P^d \) as well as \( P^e \) with \( P^f \), in which \( n \geq 2 \).

**MAIN RESULTS**

The corona product with respect to two graphs \( G \) as well as \( H \) expressed by \( G \bigcirc H \), denotes the graph yielded by employing one copy of \( G \) (with \( n \) vertices) and \( n \) copies \( H_1, H_2,..., H_n \) of \( H \). Then, the \( G \)'s \( i \)-th vertex is joined to each vertex in \( H_i \).

The corona product \( P^a \bigcirc P^c \) denotes a graph having the vertex set \( V(P^a \bigcirc P^c) = \{x, y; 1 \leq i \leq j \leq m\} \) as well as edge set \( E(P^a \bigcirc P^c) = \{x, y; 1 \leq i \leq n - 1 \} \cup \{x, y; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{y, y'; 1 \leq i \leq n, 1 \leq j \leq m - 1\} \).

Below, we consider the exact value with respect to edge irregularity strength for \( P^a \bigcirc P^c \) for \( n \geq 2 \) and \( m = 3,4,5 \).

**Lemma 2.** For any integer \( n \geq 2 \), \( \text{es}(P^a \bigcirc P^c) = 3n+1 \).
**Proof.** Assume that $P_n \ast P_j$ is a graph having a vertex set denoted by $V(P_n \ast P_j) = \{x_i, y_i^j: 1 \leq i \leq n, 1 \leq j \leq 3\}$, and as the edge set $E(P_n \ast P_j) = \{x_i, y_i^j: 1 \leq i \leq n, 1 \leq j \leq 3\}$, $1 \leq i \leq n, 1 \leq j \leq 2$.

According to Theorem 1, we have that $\text{es}(P_n \ast P_j) \geq 3n$. Since every edge $E(P_n \ast P_j) \setminus \{x_i, x_{i+1}\}$ for $1 \leq i \leq n-1$ denote a portion of complete graph $K_3$, with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Thus, the smallest edge weight 2 as well as the largest edge weight 6 will be of edges $x_i x_{i+1}$. For this there will be two pair of adjacent vertices, for instance one pair of adjacent vertices assigned label 1, a second pair of adjacent vertices assigned label $3n$, then there will be two distinct edges having the same weight. Therefore $\text{es}(P_n \ast P_j) \geq 3n+1$. To show that $\text{es}(P_n \ast P_j) \leq 3n+1$, we now express a vertex labeling $\phi_1(P_n \ast P_j) \rightarrow \{1, 2, \ldots, 3n+1\}$ as follows:

$$
\phi_1(x_i) = 2\left(\frac{i-1}{2}\right) + 1 \quad \text{if} \quad 1 \leq i \leq n
$$

and

$$
\phi_1(y_i^j) = \begin{cases} 
3(i-1) + j + \frac{j-1}{2}, & \text{if } i \text{ is odd, } 1 \leq j \leq 3 \\
3i + j + \frac{j-1}{2}, & \text{if } i \text{ is even, } 1 \leq j \leq 3
\end{cases}
$$

since

$$w_{\phi_1}(x_i x_{i+1}) = \phi_1(x_i) + \phi_1(x_{i+1}) = 6i + 2 \quad \text{for } 1 \leq i \leq n-1,$$

$$w_{\phi_1}(x_i y_i^j) = \{5i + 2\left[\frac{i-1}{2}\right] + \left[j-1\right] + j - 2, \quad \text{if } i \text{ is odd, } 1 \leq j \leq 3 \}
\quad 5i + 2\left[\frac{i-1}{2}\right] + \left[j\right] + j - 2, \quad \text{if } i \text{ is even, } 1 \leq j \leq 3$$

and

$$w_{\phi_1}(y_i^j y_i^{j+1}) = \{6(i-1) + 3j, \quad \text{if } i \text{ is odd, } 1 \leq j \leq 2 \}
\quad 6i + 3j - 5, \quad \text{if } i \text{ is even, } 1 \leq j \leq 2$$

Thus, the edge weights are distinct with respect to all pairs of different edges. Hence, the vertex labeling $\phi_1$ denotes an optimal edge irregular $(3n+1)$-labeling, which then fullfills the proof.

**Example 1.** In Figure 1, we present the $\text{es}$ labeling for graph $P_n \ast P_j$ with vertex labels and edge weight for the case $n=5$.

![Figure 1: Graph $P_n \ast P_j$ with es labelling](image-url)
Lemma 3. For any integer \( n \geq 2 \), \( es(P_n \circ P_4) = 4n + 1 \).

Proof. Let \( P_n \circ P_4 \) express a graph having vertex set given by \( V(P_n \circ P_4) = \{x_i y'_i : 1 \leq i \leq n, 1 \leq j \leq 4\} \) and the edge set \( E(P_n \circ P_4) = \{x_i x_{i+1}, 1 \leq i \leq n - 1\} \cup \{x_i y'_i : 1 \leq i \leq n, 1 \leq j \leq 4\} \cup \{y'_i y'_j : 1 \leq i \leq n, 1 \leq j \leq 3\} \).

According to Theorem 1, we have that \( es(P_n \circ P_4) \geq 4n \). Since every edge \( E(P_n \circ P_4) \) \( \{x_i x_{i+1}\} \) for \( 1 \leq i \leq n - 1 \) denote a portion of complete graph \( K_3 \), with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Therefore, the smallest edge weight 2 and the largest edge weight \( 8n \) will be of edges \( x_i x_{i+1} \). For this there will be two pairs of adjacent vertices such that one pair of adjacent vertices assign label 1, a second pair of adjacent vertices assign label \( 4n \), then there will be two distinct edges having the same weight. Therefore \( es(P_n \circ P_4) \geq 4n + 1 \). To show that \( es(P_n \circ P_4) \leq 4n + 1 \), we define a vertex labeling \( \phi_2 (P_n \circ P_4) \to \{1, 2, \ldots, 4n + 1\} \) as given below:

\[
\phi_2 (x_i) = 4i - 1, \text{ if } 1 \leq i \leq n
\]

and

\[
\phi_2 (y'_i) = \begin{cases} 4(i - 1) + j, & \text{if } 1 \leq i \leq n \text{ and } 1 \leq j \leq 24i + j - 3, \\ 4(i - 1) + j, & \text{if } 1 \leq i \leq n \text{ and } 3 \leq j \leq 4 \\
\end{cases}
\]

since

\[
w_{\phi_2} (x_i x_{i+1}) = \phi_2 (x_i) + \phi_2 (x_{i+1}) = 2(4i + 1) \text{ for } 1 \leq i \leq n - 1,
\]

\[
w_{\phi_2} (x_i y'_i) = \begin{cases} 8i + j - 5, & \text{if } 1 \leq i \leq n \text{ and } 1 \leq j \leq 28i + j - 4, \\ \leq n \text{ and } 3 \leq j \leq 4 \\
\end{cases}
\]

and

\[
w_{\phi_2} (y'_i y'_j) = \begin{cases} 8i - 5, & \text{if } 1 \leq i \leq n \text{ and } j = 18i - 2, \\ 28i + 1, & \text{if } 1 \leq i \leq n \text{ and } j = 3.
\end{cases}
\]

Thus, the edge weights are distinct with respect to all pairs of different edges. Therefore, the vertex labeling \( \phi_2 \) denotes an optimal edge irregular \((4n + 1)\)-labeling, which then completes the proof.

Example 2. In Figure 2, we present the \( es \) labelling for graph \((P_4 \circ P_4)\) with vertex labels for the case \( n = 4 \).

![Figure 2: Graph \((P_4 \circ P_4)\) with \( es \) labelling](image-url)
Lemma 4. For any integer \( n \geq 2 \), \( es(P_n \diamond P_5) = 5n+1 \).

Proof. Let \( P_n \diamond P_5 \) express a graph having the vertex set \( V(P_n \diamond P_5) = \{x_i, y_i′ : 1 \leq i \leq n, 1 \leq j \leq 5 \} \) and the edge set \( E(P_n \diamond P_5) = \{x_i, x_{i+1} ; 1 \leq i \leq n-1 \} \cup \{y_i, y_{i+1} : 1 \leq i \leq n, 1 \leq j \leq 4 \} \). According to Theorem 1, we have that \( es(P_n \diamond P_5) \geq 5 \). Since every edge \( E(P_n \diamond P_5) \) \( \{x_i, x_{i+1} \} \) for \( 1 \leq i \leq n-1 \) represent a portion of complete graph \( k_3 \), then, with respect to every edge irregular labeling, the smallest edge weight must be at least 3 of said edges. Therefore, the smallest edge weight 2 and the largest edge weight 10\( n \) will be of edges \( x_i, x_{i+1} \). For this there will be two pairs of adjacent vertices such that one pair of adjacent vertices assign label 1, second pair of adjacent vertices assign label 5\( n \), then there will be two distinct edges having the same weight. Therefore \( es(P_n \diamond P_5) \geq 5n+1 \). To show that \( es(P_n \diamond P_5) \leq 5n+1 \), we define a vertex labeling \( \phi_3(P_n \diamond P_5) \rightarrow \{1, 2, \ldots, 5n+1 \} \) as given below:

\[
\phi_3(x_i) = 2 \left( 2i + \left( \frac{i-1}{2} \right) \right), \quad \text{if } 1 \leq i \leq n
\]

and

\[
\phi_3(y_i′) = \begin{cases} 5i - j - 2, & \text{for odd } i, \\ j = 1, 2 & 5i - 2, \quad \text{for odd } i, \\ j = 3 & 5i + j - 4, \quad \text{for odd } i, \\ j = 4, 5 & 5i + j - 5, \quad \text{for even } i, \\ j = 1, 2 & 5i - 1, \quad \text{for even } i, \\ j = 3 & 5i + 5 - j, \quad \text{for even } i, \\ j = 4, 5 & \end{cases}
\]

since

\[
w_{\phi_3}(x_i, x_{i+1}) = \phi_3(x_i) + \phi_3(x_{i+1}) = 10i + 2 \quad \text{for } 1 \leq i \leq n - 1,
\]

\[
w_{\phi_3}(x_i, y_i′) = \begin{cases} 9i - j + 2 \left( \frac{i-1}{2} \right) - 2, & \text{for odd } i, \\ j = 1, 2 & 9i + 2 \left( \frac{i-1}{2} \right) - 2, \quad \text{for odd } i, \\ j = 3 & 9i + j + 2 \left( \frac{i-1}{2} \right) - 4, \quad \text{for odd } i, \\ j = 4, 5 & 9i + j + 2 \left( \frac{i-1}{2} \right) - 5, \quad \text{for even } i, \\ j = 1, 2 & 9i + 2 \left( \frac{i-1}{2} \right) - 1, \quad \text{for even } i, \\ j = 3 & 9i + 2 \left( \frac{i-1}{2} \right) - j + 5, \quad \text{for even } i, \\ j = 4, 5 & \end{cases}
\]

and

\[
w_{\phi_3}(y_i′, y_{i+1}′) = \begin{cases} 10i - 7, & \text{for odd } i, \\ j = 1, 10i - 6, & \text{for odd } i, \\ j = 2 & 10i - 2, \quad \text{for odd } i, \\ j = 3 & 10i + 1, \quad \text{for odd } i, \\ j = 4 & 10i - 7, \quad \text{for even } i, \\ j = 1, 10i - 4, \quad \text{for even } i, \\ j = 2 & 10i, \quad \text{for even } i, \\ j = 3 & 10i + 1, \quad \text{for even } i, \\ j = 4 & \end{cases}
\]
Thus, the edge weights are distinct with respect to all pairs of different edges. Hence, the vertex labeling $\phi_3$ refers to an optimal edge irregular $(5n+1)$-labeling, which fulfills the proof.

**Example 2.** In Figure 3, we present the $es$ labelling for graph $(P_4 \circ P_5)$ with vertex labels for the case $n=4$.

![Figure 3: Graph $(P_4 \circ P_5)$ with es labelling](image)

The following main result follows immediately from Lemmas 2, 3 and 4.

**Theorem 5.** For any real number $n \geq 2$ and $m = 3, 4, 4$, $es(P_n \circ P_m) = 5n+1$.

**CONCLUSION**

This study presents the exact values for edge irregularity strength with respect to corona graphs for path $P_n$ with $P_3$, $P_n$ with $P_4$ and $P_n$ with $P_5$. Recently, the case of corona graphs of $P_n$ with $P_6$ was done by Alrawajfeh *et al.* [14]. For the next research, we propose to work on some generalizations concerning the estimation of upper bound or determination of the exact value with respect to the edge irregularity strength of corona graphs of $P_n$ with $P_m$ for any $n, m \geq 2$.

**CONFLICTS OF INTEREST**

No conflict of interest is declared by the authors.

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**REFERENCES**


