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AUTOMATIC QUADRATURE SCHEME FOR CAUCHY TYPE SINGULAR INTEGRAL ON THE VARIABLE INTERVAL

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ABSTRACT

In this note, we consider the product indefinite integral of the form

$$C(h,c)=\frac{1}{\pi}\int_y^z \frac{h(t)}{t-c}dt,\ c\in(-1,1),\ \ -1\le y<z\le 1.$$

An automatic quadrature scheme (AQS) is constructed for evaluating Cauchy principal singular integrals in two cases. In the first case $c\in[y,z]\subset[-1,1]$ where $-1<y<z<1$, density function $h(t)$ is approximated by the truncated sum of Chebyshev polynomials of the first kind. Direct substitution does not give solutions so we have used the AQS and reduced problems into algebraic equation with unknown parameters b_k which can be found in terms of the singular point with some front conditions. In the second case $c\in[-1,1]$, the application of the AQS reduced the number of calculations twice and accuracy is increased. As a theoretical result, the convergence theorem of the proposed method is proven in a Hilbert space. Numerical examples with exact solutions and comparisons with other methods are also given, and they are in the line with theoretical findings.

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INTRODUCTION

One of the most valuable mathematical tools is the subject of singular integrals of order one and two in both pure and applied mathematics.

$$I_r(h,c,y,z)=\frac{1}{\pi}\int_y^z \frac{w(t)h(t)}{(t-c)^r}dt,\ c\in(-1,1),\ \ r=\{1,2\},\ -1\le y<z\le 1$$

where $w(t)$ is a weight function.

There are many studies on the numerical evaluation of singular integrals (SIs) (1) for r=1, [1]-[15] and hyper-singular integrals (HSIs) (1) at $r=\{2\}$, [16]-[27] and literatures cited therein when limit of integration $[y,z]=[a,b]$ is fixed. However not many research papers have been published if the limit of integration of (1) is not fixed. A precise evaluation of SIs and HSIs is possible only in rare cases ([4], [16]), therefore there is a need to enrich the approximate methods for evaluating SIs and HSIs. Special attention has been paid to the numerical evaluation of Cauchy principal value integrals (CPVIs) in the middle of the 19th century.

Among the researchers looking into the singularity problem were Stewart [1], Paget and Elliott [2], Chawla and Ramakrishnan [3], Ioakimidis and Theocaris [4], Monegato [5], Hasegava *et al.* ([6 -8], [10]), Boykov [9], Eshkuvatov et al. ([11-12]), Shodimetov *et al.* ([13], [15]), Nhan [14] and so on.

In this note, we consider the product indefinite integral (1) in the case of $r=\{1\}$, $w(t)=1$ and have constructed quadrature formulas called automatic quadrature scheme (AQS) based on Chebyshev polynomial approximation of the first kind for two cases: First case $c\in[y,z]\subset[-1,1]$ and second case $c\in[-1,1]$. Error estimation is obtained in the class of function $L_2[-1,1]$. Numerical solutions of AQS are compared with other methods and it is shown that the developed method is dominant over other methods and in line with theoretical findings. Particularly, in Section 2, the mathematical concepts of the first kind of Chebyshev polynomials and the detail construction of automatic quadrature scheme (AQS) for Eq. (1) are demonstrated. Convergence of the suggested method in $L_2[-1,1]$ class is presented in Section 3. Numerical examples with comparisons with other methods are given in Section 4, followed by Conclusion and Acknowledgement in Section 5.

PRELIMINARIES AND CONSTRUCTION OF AQS FOR CAUCHY TYPE SINGULAR INTEGRALS

Preliminaries

To understand the problem, let us recall the definition of Cauchy integral. The concept of one-dimensional Cauchy principal value integral (CPVI) is well understood and widely used in applications.

Definition 1 (Monegato [4])**:** In the case of a bounded interval, the standard definition of CPV integral is

$$C.P.\int_a^b \frac{h(t)}{t-c}dt = \lim_{\varepsilon\to 0}\left\{\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^{b}\right\}\frac{h(t)}{t-c}dt,\ a<c<b$$

whenever this limit exists or an alternative, or equivalent definition of CPVI is as follows

$$C.P.\int_a^b \frac{h(t)}{t-c}dt = \int_a^b \frac{h(t)-h(c)}{t-c}dt + h(c)\ln\frac{b-c}{c-a},$$

It is known that truncated Chebyshev series of the first kind is defined as

$$h(t)\approx P_N(t) = \sum_{k=0}^{N}{}' a_k T_k(t),\ -1\le t\le 1,$$

where prime ' means first term is half and $T_j(t)=\cos(j\theta)$, $t=\cos\theta$.

It is known that the coefficients a_k in (3) are defined by imposing interpolation conditions and it has exact form in the discrete case

$$a_k = \frac{2}{N}\sum_{j=0}^{N}{}'' h\left(\cos\frac{\pi j}{N}\right)\cos\left(\frac{\pi kj}{N}\right),\ 0\le k\le N,$$

where double prime " means the first and last terms are halved and in continuous case

$$a_k = \frac{2}{\pi}\int_{-1}^{1}\frac{h(t)T_k(t)}{\sqrt{1-t^2}}dt$$

The nodes are chosen as the zeros of the polynomial $\omega_{N+1}(t)$ defined by

$$\omega_{N+1}(t) = 2(t^2-1)U_{N-1}(t)$$

which yields $t_j = \cos\left(\frac{\pi j}{N}\right), j = 0,\ldots,N$ and $U_j(t)=\sin((j+1)\theta)/\sin\theta$, $t = \cos\theta$.

On the other hand, $T_j(t)$ satisfies the following orthogonality property

$$\frac{1}{\pi}\langle T_i, T_j\rangle = \frac{1}{\pi}\int_{-1}^{1}\frac{T_i(t)T_j(t)}{\sqrt{1-t^2}}dt = \begin{cases} 0, & i \neq j, \\ 1, & i = j = 0, \\ 1/2, & i = j \neq 0, \end{cases}$$

with three-term recurrence relations Mason and Handscomb ([28])

$$\begin{cases} T_m(x) = 2xT_{m-1}(x) - T_{m-2}(x), m \geq 2, \\ T_0(x) = 1,\ T_1(x) = x, \end{cases}$$

together with

$$\frac{d}{dx}T_m(x) = mU_{m-1}(x),$$

where $U_m(x)$ is a Chebyshev polynomial of the second kind.

Construction of AQS for Cauchy Type Singular Integrals

Let us consider, the Cauchy type singular integrals on the variable intervals:

$$C(h,c,y,z) = \frac{1}{\pi}\int_y^z \frac{h(t)}{t-c}dt,\ c\in(-1,1),\ -1 \leq y < z \leq 1,$$

and construct AQS for (10). To do this end we consider two cases:

Case 1: Let $c\in[x,y]\subset[-1,1]$, then replacing h with truncated sum of Chebyshev polynomials $P_N(t)$ defined by (3), we obtain

$$\begin{aligned} C(h,c,y,z) &= \frac{1}{\pi}\left[\int_y^z \frac{h(t)-h(c)}{t-c}dt + h(c)\int_y^z \frac{dt}{t-c}\right] \\ &= \frac{1}{\pi}\left[\int_y^z \frac{P_N(t)-P_N(c)}{t-c}dt + h(c)\int_y^z \frac{dt}{t-c}\right] + R_{N,r}(h,c,y,z) \\ &= Q_N(h,c,y,z) + R_N(h,c,y,z), \end{aligned}$$

where $R_{N,r}(h,c,y,z)=C(h,y,z,c)-Q_N(h,y,z,c)$ is error bound and

$$Q_N\left(h,c,y,z\right)=\frac{1}{\pi}\left[\int_y^z\frac{P_N\left(t\right)-P_N\left(c\right)}{t-c}dt+h\left(c\right)\int_y^z\frac{dt}{t-c}\right]$$
$$=J_1\left(c,y,z\right)+J_2\left(c,y,z\right),$$

with

$$J_1\left(c,y,z\right)=\frac{1}{\pi}\int_y^z\frac{P_N\left(t\right)-P_N\left(c\right)}{\left(t-c\right)}dt,\ J_2\left(c,y,z\right)=\frac{1}{\pi}h\left(c\right)\int_y^z\frac{dt}{\left(t-c\right)}.$$

Let the function $h(t)$ be approximated by a truncated sum of the Chebyshev polynomials of the first kind defined by (3) and coefficients a_k are computed by (4) or (5) depending on the existence of the integral. Evaluation of $J_1(c,y,z)$ can be obtained by the method given in Hasegawa [9]

$$\frac{P_N\left(t\right)-P_N\left(c\right)}{t-c}=\sum_{k=0}^{N-1}{}' b_kT_k\left(t\right),$$

and using well known integration formula Mason and Handscomb ([28], pp. 32)

$$\int T_n\left(t\right)dt=\begin{cases}\frac{1}{4}\left[T_2\left(t\right)+T_0\left(t\right)\right],\ n=1,\\ \frac{1}{2}\left[\frac{T_{n+1}\left(t\right)}{n+1}-\frac{T_{|n-1|}\left(t\right)}{n-1}\right],n\neq 1.\end{cases}$$

From (13)-(15), it follows that

$$J_1\left(c,y,z\right)=\frac{1}{2\pi}\left\{b_0\,T_1\left(t\right)\Big|_y^z+\frac{1}{2}b_1\left[T_2\left(t\right)+T_0\left(t\right)\right]\Big|_y^z\right.$$
$$\left.+\sum_{k=2}^{N-1}b_k\left[\frac{T_{k+1}\left(t\right)}{k+1}-\frac{T_{k-1}\left(t\right)}{k-1}\right]_y^z\right\}=\frac{1}{\pi}\sum_{k=0}^{N}B_k\,T_k\left(t\right)\Big|_y^z,$$

where

$$B_0=\frac{b_1}{4},\ B_k=\frac{b_{k-1}-b_{k+1}}{2k},\ k=\left\{1,2,\ldots,N\right\},\ b_N=b_{N+1}=0.$$

Evaluation of $J_2(c,y,z)$ is straightforward

$$J_2\left(c,y,z\right)=\frac{1}{\pi}h\left(c\right)\ln\frac{z-c}{c-y}.$$

Substituting (16) and (18) into (12) yields

$$Q_N\left(h,c,y,z\right)=\frac{1}{\pi}\left[\frac{1}{2}\sum_{k=1}^{N}B_k\left[T_k\left(z\right)-T_k\left(y\right)\right]+h\left(c\right)\ln\frac{z-c}{c-y}\right],$$

where coefficients B_k are defined by (17).

To find the coefficients b_k rewrite Eq. (14) in the form

$$P_N(t)-P_N(c)=\sum_{k=0}^{N-1}{}' b_k T_k(t)(t-c).$$

It is known that

$$tT_k(t)=\frac{1}{2}\left[T_{k+1}(t)+T_{k-1}(t)\right],\ k=1,\ldots,N-1.$$

Using (21) and restructuring the right-hand side of (20), we arrive at

$$\sum_{k=1}^{N} a_k\left(T_k(t)-T_k(c)\right)=+\frac{1}{2}(b_1-b_0c)T_0(t)+\frac{1}{2}\sum_{k=1}^{N}(b_{k-1}-2cb_k+b_{k+1})T_k(t).$$

with $b_N = b_{N+1} = 0$ and relationship (22) is valid for any $t \in [-1,1]$, particularly for $t = c$ gives

$$0=\frac{1}{2}\sum_{k=1}^{N}(b_{k-1}-2cb_k+b_{k+1})T_k(c)+\frac{1}{2}(b_1-b_0c)T_0(c).$$

Subtracting the zero term given by (23) on th right-hand side of (22), we obtain

$$\sum_{k=1}^{N} a_k\left(T_k(t)-T_k(c)\right)=\frac{1}{2}\sum_{k=1}^{N}(b_{k-1}-2cb_k+b_{k+1})\left(T_k(t)-T_k(c)\right).$$

By equating both sides of (24), we arrive at three diagonal system of algebraic equation

$$\begin{aligned}&b_{k-1}-2cb_k+b_{k+1}=2a_k,\quad k=1,2,\ldots N.\\&b_N=b_{N+1}=0.\end{aligned}$$

Double sweep method is used to find the coefficients $b_k = 0, k = 0, K, N-1$ in the backward direction with starting $b_N = b_{N+1} = 0$. Once we find the value of $b_k = b_k = (c)$ then $B_k = B_k(c)$ can easily computed through the (17) and approximate solutions of the problem is computed by (19).

Case 2: Let $[y,z]=[-1,1]$ and $c \in (-1,1)$. In this case, density function $h(t)$ is approximated by the truncated sum of Chebyshev series of the first kind defined by (3) and we obtain

$$\begin{aligned}C_0(h,c)&=\frac{1}{\pi}\int_{-1}^{1}\frac{h(t)}{t-c}dt=\frac{1}{\pi}\int_{-1}^{1}\frac{h(t)-h(c)}{t-c}dt+\frac{h(c)}{\pi}\int_{-1}^{1}\frac{dt}{t-c}\\&=\frac{1}{\pi}\int_{-1}^{1}\frac{P_N(t)-P_N(c)}{t-c}dt+\frac{h(c)}{\pi}\int_{-1}^{1}\frac{dt}{t-c}+R_N(h,c)\\&=Q_{N0}(h,c)+R_{N0}(h,c),\end{aligned}$$

where

$$Q_{N0}(h,c) = \frac{1}{\pi}\int_{-1}^{1}\frac{P_N(t)-P_N(c)}{t-c}dt + \frac{h(c)}{\pi}\int_{-1}^{1}\frac{dt}{t-c}$$
$$= I_1(h,c) + I_2(h,c),$$

Second integral in (27), can be easily integrated

$$I_2(h,c) = \frac{h(c)}{\pi}\ln\left|\frac{1-c}{1+c}\right|$$

For the first integral in (27) we applied Hasegawa [9] approach, as well as used (14)-(15) to yield

$$I_1(h,c) = \frac{1}{\pi}\left[\frac{1}{2}b_0\int_{-1}^{1}T_0(t)dt + b_1\int_{-1}^{1}T_1(t)dt + \sum_{k=2}^{N-1}b_k\int_{-1}^{1}T_k(t)dt\right]$$
$$= \frac{1}{\pi}\left|b_0 - \sum_{k=1}^{\left\lceil\frac{N+1}{2}\right\rceil-1} b_{2k}\frac{2}{(2k)^2-1}\right|,$$

where coefficients b_k are defined by (25).

AQS is obtained by substituting Eq. (28) and (29) into (27)

$$Q_{N0}(h,c) = \frac{1}{\pi}\left[b_0(c) - \sum_{k=1}^{\left\lceil\frac{N+1}{2}\right\rceil-1} b_{2k}(c)\frac{2}{(2k)^2-1} + h(c)\ln\left|\frac{1-c}{1+c}\right|\right].$$

From (30), it follows that in the case of the process of calculation is reduced by half of the summation and it helps us to economize the computational time.

ERROR ESTIMATION OF THE AQS FOR CPSIS

In this section, the error estimation of the proposed AQS is obtained in the different classes of functions. Before proving the main theorems, we need some preliminaries. The following theorem shows that if $h(t)$ is a smooth function, the truncated series of the Chebyshev polynomials defined by (3) converges to $h(t)$ very rapidly as N goes towards infinity.

Theorem 1: (Mason and Handscomb [28], pp. 131]: If the function $h(t)$ has m+1 continuous derivatives on [-1,1] then $|h(t)-P_N(t)| = O(n^{-m})$, for all $t \in [-1,1]$.

Theorem 2: (Mason and Handscomb [28]): If $h(x)$ is continuous and either of bounded variation or satisfying a Dini–Lipschitz condition on [-1,1], then its truncated series of the Chebyshev polynomials of the first kind is uniformly convergent

$$\|h - P_n\| \le w(\delta) \xrightarrow[\delta\to 0]{} 0,$$

where $w_f(\delta) = \sup_{x,y\in[-1,1],|x-y|\le\delta}\{|f(x)-f(y)|\}$ is the modulus of continuity of the function.

Lemma 1: (Mason and Handscomb [28]): If $p \geq 1$, $q \geq 1$ and $1/p+1/q=1$ and if $h(t)$ is L_{2}- integrable over the interval [-1,1] with the weight function $\rho(t)$, then

$$|\langle f,g\rangle| \leq \|f\|_p \|g\|_q = \left(\int_{-1}^{1} |f(t)|^p \rho(t)dt\right)^{1/p} \left(\int_{-1}^{1} |g(t)|^q \rho(t)dt\right)^{1/q}.$$

Lemma 2: (Generalized Holder's inequality), (Sobolev [29], pp.13): Let $\rho(x)$ be a weight function and $\lambda_1>1, i=1,2,3$ with $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=1$, then

$$\left|\int_a^b u(x)v(x)h(x)\rho(x)dx\right| \leq \left(\int_a^b |u(x)|^{\lambda_1} \rho(x)dx\right)^{\frac{1}{\lambda_1}} \left(\int_a^b |v(x)|^{\lambda_2} \rho(x)dx\right)^{\frac{1}{\lambda_2}} \left(\int_a^b |h(x)|^{\lambda_3} \rho(x)dx\right)^{\frac{1}{\lambda_3}}.$$

Let residual term of given function $h(t)$ be given

$$e_N(t) = h(t) - P_N(t), \ \|e_N(t)\| = \max_{t\in[y,z]\subseteq[-1,1]} |h(t) - P_N(t)|.$$

where $P_N(t)$ be defined by (3).

Let us formulate the main theorem.

Theorem 3: Let $h' \in L_{2\rho}[-1,1]$ and $P_N(t)$ be the interpolating polynomial at the zeros of $\omega_{N+1}(t)=2(t^2-1)$ $U_{N-1}(t)$ defined by (6). Then the constructed AQS $Q_{N0}(h,c)$ defined by (30) converges to original singular integral $C_0(h,c)$ defined by (26) in the sense of L_q norm

$$\|R_{N0}(h,c)\|_q = \|C_0(h,c) - Q_{N0}(h,c)\|_q \xrightarrow[N\to\infty]{} 0$$

where $1 < q < \frac{p}{p_0}$. and $p>p_o>1$.

Proof: From the generalized Holder inequality (33), let $\rho(x)=1$ and

$$\lambda_1 = q, \lambda_2 = \frac{1}{1/p - 1/q}, \lambda_3 = p' = \frac{p}{p-1}.$$

Due to (11), the error bound of $R_{N0}(h,c)$ in (26) is estimated as follows:

$$|R_{N0}(h,c)| = |C_0(h,c) - Q_{N0}(h,c)| = \frac{1}{\pi}\left|\int_{-1}^{1} \frac{e_N(t)-e_N(c)}{t-c}dt\right|$$

Case 1: Let $t \to c$ then

$$|R_{N0}(h,c)| \leq \|e'_N(t)\| \left|\int_{-1}^{1} dt\right| = 2\|h'(t) - P'_N(t)\| \xrightarrow[N\to\infty]{} 0$$

Case 2: For any $C\in(-1,1)$, $t \neq c$ we have

$$\left|R_{N0}(h,c)\right| = \left|\int_{-}^{1}\frac{e_N(t)-e_N(c)}{t-c}dt\right| = \left|\int_{-1}^{1}\left(\frac{\left[e_N^*(t)\right]^p}{(t-c)^\alpha}\right)^{1/q}\left(e_N^*(t)\right)^{1-p/q}\left(\frac{1}{t-c}\right)^{1-\alpha/q}dt\right|$$

$$\leq \left(\int_{-1}^{1}\frac{\left|e_N^*(t)\right|^p}{|t-c|^\alpha}dt\right)^{1/q}\left(\int_{-1}^{1}\left|e_N^*(t)\right|^{(1-p/q)\frac{1}{1/p-1/q}}dt\right)^{1/p-1/q}\left(\int_{-1}^{1}\left(\frac{1}{|t-c|}\right)^{p'(1-\alpha/q)}dt\right)^{1/p'}$$

$$= I_1(c)I_2(c)I_3(c), \qquad ,$$

where $e^*_N(t) = e_N(t)\text{-}e_N(c)$ and

$$I_1(c) = \left(\int_{-1}^{1}\frac{\left|e_N^*(t)\right|^p}{|t-c|^\alpha}dt\right)^{1/q}, I_2(c) = \left(\int_{-1}^{1}\left|e_N^*(t)\right|^{(1-p/q)\frac{1}{1/p-1/q}}dt\right)^{1/p-1/q}, I_3(c) = \left(\int_{-1}^{1}\left(\frac{1}{|t-c|}\right)^{p'(1-\alpha/q)}dt\right)^{1/p'}.$$

Since $0 < \alpha < 1$ and $e^*_N(t)$ is a smooth function, then $I_1(c)$ exists as an improper integral, whereas $I_2(c)$ can be computed as

$$I_2(c) = \left(\int_{-1}^{1}\left|e_N^*(t)\right|^p dt\right)^{1/p(1-p/q)} = \left\|e_N^*\right\|_p^{1-p/q},$$

where

$$\left\|e_N^*\right\|_p = \left(\int_{-1}^{1}\left|\left(e_N^*\right)^p\right|dt\right)^{1/p}.$$

For $I_3(c)$, we assume that $0 < \frac{1}{p_o} \leq \alpha < 1$. Then since we have $\frac{1}{p_o} > \frac{q}{p}$,

$$\frac{\alpha}{q} > \frac{1}{p} \text{ or } 1-\frac{\alpha}{q} < 1-\frac{1}{p} = \frac{1}{p'},$$

which yields $p'\left(1-\frac{\alpha}{q}\right) < 1$. Thus the integral $I_3(c)$ exists and bounded for all $c\in(-1,1)$, i.e.

$$I_3(c) = \left(\int_{-1}^{1}\frac{dt}{|t-c|^{p'(1-\alpha/q)}}\right)^{1/p'} = \left\|\frac{1}{(t-c)^{(1-\alpha/q)}}\right\|_{p'} \leq M_q < \infty.$$

Substituting $I_1(c)$, $I_2(c)$ and $I_3(c)$ into (36) we obtain

$$\left|R_N(h,c)\right| \leq \left(\int_{-1}^{1}\frac{\left|e_N^*(t)\right|^p dt}{|t-c|^\alpha}\right)^{1/q}\left\|e_N^*\right\|_p^{(1-p/q)}M_{pq}.$$

Raising both sides of (38) to the *q-th* power and integrating with respect to *c*, yields

$$\|R_N\|_q^q = \int_{-1}^{1} \left|R_N(h,c)\right|^q dc \le M_{pq}^q \|e_N^*\|_p^{q-p} \int_{-1}^{1} \left(\int_{-1}^{1} \frac{\left|e_N^*(t)\right|^p}{|t-c|^\alpha} dt \right) dc$$

$$= M_{pq}^q \|e_N^*\|_p^{q-p} \int_{-1}^{1} \left|e_N^*(t)\right|^p dt \int_{-1}^{1} \frac{dc}{|t-c|^\alpha} = M_{pq}^q B \|e_N^*\|_p^q,$$

where $B = \max_{-1\le t\le 1} \left| \int_{-1}^{1} \frac{dc}{|t-c|^\alpha} \right|$ for a chosen α. We obtain

$$\|R_N\|_q \le M_{pq} \sqrt[q]{B} \|e_N^*\|_p \xrightarrow[N\to\infty]{} 0.$$

This completes the proof of the Theorem 3.

NUMERICAL EXPERIMENTS

In this section, we consider three examples and show that developed AQS is highly accurate and in some cases gives the exact solution.

Example 1: Consider the following singular integral of Cauchy type

$$C_0(h,c) = \frac{1}{\pi} \int_{-1}^{1} \frac{t^3 - 3 + 10}{t-c} dt, \; c \in (-1,1),$$

The analytical solution of (40) is

$$C_0(h,c) = \frac{1}{\pi} \left[\frac{2}{3} + 2\left(c^2 + 2c\right) + \left(c^3 - 3c + 10\right) \ln \frac{1-c}{1+c} \right]$$

Difference between exact and approximate solution for different values of N (number of terms in Chebyshev series) in the case of $y = -1$, $z = 1$and c∈(-1,1) is given in Table 1.

Table 1: The case: c∈(-1,1)

			N=2	N=3	N=5	N=10
c	**Exact sol**	**AQS (29)**	**Abs Error**	**Abs Error**	**Abs Error**	**Abs Error**
-0.99	19.145	18.786	0.359	0.0	0.0	0.0
-0.79	6.801	6.669	0.132	0.0	0.0	0.0
-0.45	1.905	2.042	-0.136	0.0	0.0	0.0
-0.12	-0.893	-0.637	-0.256	0.0	0.0	0.0
0	-1.698	-1.432	-0.265	0.0	0.0	0.0
0.12	-2.429	-2.173	-0.256	0.0	0.0	0.0
0.45	-4.266	-4.130	-0.136	0.0	0.0	0.0
0.79	-6.841	-6.973	0.132	0.0	0.0	0.0
0.99	-14.553	-14.912	0.359	0.0	0.0	0.0

Example 2: Consider the following singular integral of Cauchy type

$$C_0(h,c) = \frac{1}{\pi}\int_{-0.89}^{0.89} \frac{t^3 - 3t + 10}{t - c}\,dt,\ c \in (-0.89, 0.89),$$

the exact solution of (42) is

$$C_0(h,c) = \frac{1}{\pi}\left[\frac{2\cdot(0.89)^3}{3} + 1.78\cdot\left(c^2 + 2c\right) + \left(c^3 + 2c^2 - 1\right)\ln\frac{0.89 - c}{0.89 + c}\right].$$

Numerical results of exact solution (43) and residual of AQS are presented in Table 2.1 and 2.2 for different value of singular points $c \in (-0.89, 0.89)$ and $c \in (-1, -0.89) \cup (0.89, 1)$

Table 2.1: The case c∈(-0.89,0.89)

c		***N=2***	***N=2***	***N=3***	***N=5***	***N=10***
	Exact (41)	**AQS (18)**	**Error term**	**Error term**	**Error term**	**Error term**
-0.889	11.038	10.866	0.172	0.0	0.0	0.0
-0.789	7.641	7.563	0.077	0.0	0.0	0.0
-0.489	2.570	2.710	-0.140	0.0	0.0	0.0
-0.189	-0.239	0.016	-0.255	0.0	0.0	0.0
0	-1.550	-1.275	-0.275	0.0	0.0	0.0
0.189	-2.683	-2.428	-0.255	0.0	0.0	0.0
0.489	-4.401	-4.261	-0.140	0.0	0.0	0.0
0.789	-6.815	-6.892	0.077	0.0	0.0	0.0
0.889	-8.431	-8.604	0.172	0.0	0.0	0.0

Table 2.2: The case $c \in (-1, -0.89) \cup (0.89, 1)$

c		***N=2***	***N=2***	***N=3***	***N=5***	***N=10***
	Exact (41)	**AQS (18)**	**Abs Error**	**Abs Error**	**Abs Error**	**Abs Error**
-0.998	29.771	29.482	0.289	0.0	0.0	0.0
-0.958	15.836	15.592	0.245	0.0	0.0	0.0
-0.918	12.570	12.368	0.202	0.0	0.0	0.0
-0.899	11.520	11.338	0.183	0.0	0.0	0.0
0.889	-8.431	-8.604	0.172	0.0	0.0	0.0
0.918	-9.189	-9.391	0.202	0.0	0.0	0.0
0.958	-10.867	-11.112	0.245	0.0	0.0	0.0
0.998	-18.578	-18.867	0.289	0.0	0.0	0.0

Example 3: Comparisons AQS with other methods. Consider the following singular integral of Cauchy type

$$C_0\left(h,c\right)=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\left(t^2+a^2\right)\left(t-c\right)}dt,\ c\in\left(-1,1\right),a=\left\{1,\frac{1}{4},\frac{1}{8}\right\}$$

Exact solution of SIs (44) with $c\in(-1,1)$ for any $a\in R$ is given by

$$C_0\left(h,c\right)=\frac{1}{\pi}\frac{1}{c^2+a^2}\left[-\frac{2c}{a}\tan^{-1}\left(\frac{1}{a}\right)+\ln\frac{1-c}{1+c}\right].$$

Numerical results of exact solution and residual of AQS are presented in Table 3.1-3.2 for different value of singular points $c\in(-1,1)$.

Table 3.1: Proposed AQS in the case of $c\in(-1,1)$, $a=1$.

c	Exact	Abs Error, N=5	Abs Error, N=10	Abs Error, N=15	Abs Error, N=17
0.2	-0.220	0.007792390	0.000231468	0.000000039	0.000000211
0.5	-0.480	0.001358745	0.000226799	0.000001143	0.000000036
0.95	-0.863	0.003680672	0.000212502	0.000000614	0.000000047

Table 3.2: Comparisons developed AQS and developed method in [9] at three values of $c\in(-1,1)$ and $a=1$.

c	Exact	Abs Error of AQS N=17	AQS (Hasegava [9]) N=22
0.2	-0.2707	$2\text{x}10^{-7}$=0.000000211	$1\text{x}10^{-8}$=0.00000001
0.5	-0.4798	$3\text{x}10^{-7}$=0.000000036	$1\text{x}10^{-8}$=0.00000001
0.95	-0.8626	$4\text{x}10^{-8}$=0.000000047	$1\text{x}10^{-9}$=0.000000001

Table 3.1 shows the error of AQS when number of nodes N increases. In Table 3.2, shows the comparisons between AQS and the proposed method in [9]. We are not able to run our codes in MATLAB for N=22 because we lack a high-performance computer for such computational complexity. From Table 3.2 we can conclude that the proposed AQS is comparable with the method developed in [9].

CONCLUSION

In this paper, we have constructed automatic quadrature schemes based on truncated series of Chebyshev polynomials of the first kind, for the simple weighted singular integrals on the variable interval $[y,z]\subseteq[-1,1]$. We have also found theformulas for unknown discrete coefficients b_k by the use of three-term relations and front conditions. Additionally, the error estimate of AQS was obtained in the class L_2 [-1,1]. Numerical examples demonstrated the validity and accuracy of the methods and it is in line of theoretical findings.

CONFLICTS OF INTEREST

The authors declare there are no competing interests.

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