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STABILITY OF THE NUMERICAL SOLUTION FOR THE MIXED PROBLEM OF THE SAINT-VENANT EQUATIONS

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ABSTRACT

This article is devoted to the construction and study of the exponential stability of an explicit upwind difference scheme for a mixed problem for the linear system of the Saint Venant equation. For the numerical solution of the mixed problem for the linear system of the Saint Venant equation, an explicit upwind difference scheme is constructed. For a numerical solution, a discrete Lyapunov function is constructed and an a priori estimate for it is obtained. On the basis of the discrete Lyapunov function, the exponential stability of the numerical solution of the initial-boundary-value difference problem of the mixed problem for the linear system of the Saint Venant equation is proved. A theorem on the exponential stability of the numerical solution of the initial-boundary-value difference problem is proved. The behavior of the discrete Lyapunov function is numerically investigated depending on the algebraic condition of exponential stability of the numerical solution of the mixed problem. The results of the theorem on the exponential stability of the numerical solution are confirmed by a specific example of an open channel flow problem.

INTRODUCTION

It is known that the behavior of water in rivers, lakes, oceans, as well as in small bodies is described by the Saint-Venant equations [1]. The Saint-Venant equations are a system of hyperbolic partial differential equations that describe flows below the surface of a fluid.

The one-dimensional theory of unsteady flows without discontinuities, i.e., flows that are not accompanied by the formation of discontinuous waves, can be considered the most developed today (including the problem of natural flooding). The main direction of research in the field of calculating unsteady motion described by the Saint-Venant equations is the development of methods for numerical calculation that are convenient when using computer technology. An important point in

the construction of a numerical solution to the Saint-Venant equations is the stability of the difference scheme. The equations considered in the paper are differential equations of hyperbolic type, which does not always ensure the absolute stability of the method. The solution of these issues was and remains one of the crucial tasks.

The purpose of this work is to develop and prove the stability of an explicit difference scheme for determining the numerical solution of a mixed problem for the system of the Saint-Venant equations. In this paper, we use the Lyapunov method to study the exponential stability of the solution of the nonhomogeneous Saint-Venant equations with arbitrary friction and a slope that changes in space. The advantage is that when using this method, you only need to measure the value of the desired functions at

the boundary, which is much easier for practical implementations.

Currently, various methods for solving the Saint-Venant equations have been developed, such as the method of characteristics, the method of grids, the Lyapunov quadratic function method, the variational method, etc. In [2] for solving one-dimensional equations of unsteady motion in open canals, an implicit difference grid scheme was developed that allows calculations to be performed with a large time step. It was especially important for calculating floods in large rivers, when the duration of the calculated process took quite a long time. However, there is no description of the algorithm and justification of the finite difference method itself.

The study by [3] is devoted to exponential stability of non-linear Saint-Venant equations in differential form. An explicit quadratic Lyapunov function is constructed, and the local exponential stability is proved. In the book [4], the theory of symmetric hyperbolic system is described. Particularly, in the case of two spatial and one temporal variables, the existence theorem of a dissipative mixed problem is stated. Numerical calculations of simple models are given as examples.

The monograph [5] is devoted to the study of mixed problems for one-dimensional hyperbolic systems in canonical form. Lyapunov stability is established in various functional spaces, in particular, many practical models are considered. However, the problems of numerical solution for mixed problems are not considered.

In [6] for telegraph equation, the discrete Lyapunov function is constructed and its decreasing is proved. Using such approach to the Saint-Venant equations is due to the difficulties that require additional research. For two-dimensional hyperbolic equations with dissipative boundary conditions, the exponential stability of the solution is established by the Lyapunov function method in [7]. Whereas, in [8] algebraic conditions for exponential stability of the solution of mixed problems of linear Saint-Venant equations are obtained. The discussion of numerical solution is not considered.

For one-dimensional quasi-linear hyperbolic systems, problems with dissipative boundary conditions that guarantee exponential stability of classical solutions are considered in [9]. There are no studies on numerical calculations. In [10], the second kind Volterra transformation and reversible Fredholm transformation optimal management problems for general linear hyperbolic systems are investigated.

Note that in the works [3, 5, 7–10] we study the issues related to the theoretical aspects of the solvability and stability of mixed problems for hyperbolic systems and the issues of constructing numerical solutions and the stability of difference schemes are not considered. It is crucial to consider the numerical calculation of mixed problems for hyperbolic systems due to the fact that the dimension of the linear algebraic equations system increases with increasing dimension of the considered area. This leads to an unreasonably large amount of computing and requires the involvement of high-performance computing equipment.

In [11] the use of various difference grids (rectilinear and curved) for the numerical calculation of linear partial differential equations is shown. The study does not consider the problems of the Saint-Venant equations, this is apparently due to the nonlinearity of the equation.

The discretization of equations describing the unsteady flow of a viscous incompressible fluid is considered in [12], employing the finite difference method and the splitting scheme by physical factors on a rectangular non-uniform grid with a staggered arrangement of nodes. However, there is no justification for the convergence and stability of the difference scheme.

In [13], the authors proposed a class of difference schemes for hyperbolic systems of equations that have several forms of notation. The stability of the proposed difference schemes is investigated using the technique of energy integrals. However, their application to the study of exponential stability is a rather difficult task.

The work [14] is devoted to the study of initial boundary value problems for a class

of three-dimensional quasi-linear hyperbolic systems. An a priori estimate of the problem solution by means of the energy integral method is obtained. The problems of the numerical solution and its stability have not been studied. A linear initial-boundary value problem of the dynamics of fluid-saturated porous media, described by three elastic parameters in a reversible hydrodynamic approximation, is solved numerically in [15]. The issue of the computational model adequacy remains open.

In [16], a problem for the Saint-Venant-Exner equation (SVE), which describes the dynamics of water in a canal filled with sediments with arbitrary values of the canal bottom slope, friction, porosity, as well as the interaction of water and sediment under subcritical or supercritical flow regime is considered. However, this is the subject of further research. We consider the case of a canal without sediment.

The book [17] is devoted to the solution methods of high order algebraic systems, which appear when using the grid method to the problems of mathematical physics. Along with iterative methods, which are most widely used in computational practice in solving these problems, direct methods are also described. Here we upwind explicit difference scheme in relation to the Saint-Venant equations.

According to [3], we will consider a hyperbolic system with variable coefficients and with lower-order terms

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x + \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \tag{1}$$

and with boundary conditions at $x = 0, L$, respectively,

$$\begin{cases} y_1(t,0) = ry_2(t,0), \\ y_2(t,L) = sy_1(t,L), \end{cases} \forall t > 0 \tag{2}$$

and with initial data at $t = 0$:

$$\begin{cases} y_1(0,x) = y_{10}(x), \\ y_2(0,x) = y_{20}(x), \end{cases} 0 \le x \le L \tag{3}$$

where

$$\lambda_1(x) = \sqrt{gH^*} + V^* > 0, \lambda_2(x) = \sqrt{gH^*} - V^* > 0,$$

$$a(x) = \varphi(x)\delta_1(x), b(x) = \varphi^{-1}(x)\gamma_2(x),$$

$$\varphi_1(x) = \exp\left(\int_0^x \frac{\gamma_1(s)}{\lambda_1(s)}ds\right), \varphi_2(x) = \exp\left(-\int_0^x \frac{\delta_2(s)}{\lambda_2(s)}ds\right), \varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)},$$

$$\gamma_1(x) = -\frac{3f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}},$$

$$\delta_1(x) = -\frac{f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}},$$

$$\gamma_2(x) = \frac{f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}},$$

$$\delta_2(x) = \frac{3f(H^*,V^*)}{4(\sqrt{gH^*}+V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}},$$

$$f(H^*,V^*) = \frac{kV^{*2}}{H^*} - gC,$$

$$k_0 = \frac{b_0 H^*(0) + \sqrt{gH^*(0)}}{b_0 H^*(0) - \sqrt{gH^*(0)}}, k_1 = \frac{b_1 H^*(L) - \sqrt{gH^*(L)}}{b_1 H^*(L) + \sqrt{gH^*(L)}},$$

$$r = k_0\frac{\varphi_1(0)}{\varphi_2(0)}, s = k_1\frac{\varphi_2(L)}{\varphi_1(L)},$$

$$(y_{10}, y_{20})^T \in L^2((0,L); \mathbb{R}^2),$$

$$\begin{pmatrix} y_1(t,x) \\ y_2(t,x) \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\begin{pmatrix} \sqrt{\frac{g}{H^*}} & 1 \\ -\sqrt{\frac{g}{H^*}} & 1 \end{pmatrix}\begin{pmatrix} H(t,x) \\ V(t,x) \end{pmatrix},$$

where $y_1(t,x)$, $y_2(t,x)$ are unknown functions to be determined; $H(t,x) = H^*(x)$, $V(t,x) = V^*(x)$ are the stationary solution of the system of Saint-Venant equations and $H(t,x)$ is the water depth, $V(t,x)$ is the horizontal water velocity functions of two variables which is unknown to be determined. The slope $C(\bullet) \in C^2([0,L])$ is the channel bottom slope, g is the constant gravity acceleration and k is a constant friction coefficient. For a description and meaning of other parameters, see [3].

DIFFERENCE SPLITTING SCHEME

We will build a difference grid in area $G = \{(t,x): 0 \le t \le T, 0 \le x \le L\}$ with steps Δt in the direction t and Δx in the direction x. The nodal points of the difference mesh (meaning the intersection of straight lines $t = t^\kappa \triangleq \kappa \Delta t$ and $x = x_j \triangleq j\Delta x$) are denoted by (t^κ, x_j). The set of nodal points of the difference grid is denoted by G_h, where

$$G_h \triangleq \{(t^\kappa, x_j): \kappa=0,\dots,K; j=0,\dots,J\}$$

and the values of the numerical solution at the nodal points are denoted by

$$(y_i)_j^\kappa = y_i(t^\kappa, x_j), i=1,2; \kappa=0,\dots,K; j=0,\dots,J.$$

We select the steps of the difference grid Δt, Δx in such a way that the equalities $K\Delta t = T$ and $J\Delta x = L$.

To find a numerical solution for the mixed problem of equations (1)–(3) over the difference grid G_h, we propose the following upwind difference splitting scheme for the lower terms

$$\begin{cases} (z_1)_j^\kappa = (y_1)_j^\kappa - (\lambda_1)_j \frac{\Delta t}{\Delta x}\left[(y_1)_j^\kappa - (y_1)_{j-1}^\kappa\right], j = 1,\dots,J; \\ (z_2)_j^\kappa = (y_2)_j^\kappa - (\lambda_2)_j \frac{\Delta t}{\Delta x}\left[(y_2)_j^\kappa - (y_2)_{j+1}^\kappa\right], j = 0,\dots,J-1; \end{cases} \kappa = 0,\dots,K-1, \tag{4}$$

$$\begin{cases} (y_1)_j^{\kappa+1} = (z_1)_j^\kappa - \Delta t a_j (z_2)_{j-1}^\kappa, j = 1,\dots,J; \\ (y_2)_j^{\kappa+1} = (z_2)_j^\kappa - \Delta t b_j (z_1)_{j+1}^\kappa, j = 0,\dots,J-1; \end{cases} \kappa = 0,\dots,K-1. \tag{5}$$

Boundary conditions (2) are approximated in the following way,

$$\begin{cases} (y_1)_0^\kappa = s(y_2)_0^\kappa, \\ (y_2)_J^\kappa = r(y_1)_J^\kappa, \end{cases} \kappa = 1,\dots,K. \tag{6}$$

Initial conditions (3) are approximated as follows,

$$\begin{cases} (y_1)_j^0 = (y_{10})_j, \\ (y_2)_j^0 = (y_{20})_j, \end{cases} j = 0,\dots,J. \tag{7}$$

Suppose that the steps of the difference grid satisfy the Courant-Friedrichs-Levy (CFL) condition,

$$\frac{\Delta t}{\Delta x} \max_{\substack{1 \le i \le 2 \\ 1 \le j \le J}} |(\lambda_i)_j| \le 1. \tag{8}$$

Now let us investigate the question of the stability of the numerical solution for the difference problem (4) - (7). First, we give the definition of the exponential stability of the numerical solution for this difference problem.

Definition 2.1: The solution to the difference scheme (4) - (5) satisfying the boundary conditions (6) is called exponentially stable, if there are positive constants $\eta > 0$ and $c > 0$ such that for any initial condition $((y_0)_j) \triangleq ((y_{10})_j, (y_{20})_j)^T \in L^2(\{x_j\}, j=0,\dots J; \mathbb{R}^2)$, the solution of the difference initial-boundary value problem (4) - (7) satisfies the inequality

$$\Delta x \sum_{j=1}^J [(y_1)_j^\kappa]^2 + \Delta x \sum_{j=0}^{J-1} [(y_2)_j^\kappa]^2 \le ce^{-\eta t_\kappa}\left\{\Delta x \sum_{j=1}^J [(y_{10})_j]^2 + \Delta x \sum_{j=0}^{J-1} [(y_{20})_j]^2\right\}, \kappa = 1,\dots,K$$

where $L^2(\{x_j\}, j=0,\dots J; \mathbb{R}^2)$ is a discrete space L^2, the norm in which is determined by the equality

$$\|(y_0)_j\|_{L^2(\{x_j\}, j=0,\cdots J; \mathbb{R}^2)} = \sqrt{\Delta x \sum_{j=1}^J [(y_{10})_j]^2 + \Delta x \sum_{j=0}^{J-1} [(y_{20})_j]^2}$$

and the norm is limited.

Consider the difference boundary value problems (4) - (6) with a stationary solution

$$y_j^\kappa \triangleq \begin{pmatrix} [y_1]_j^\kappa \\ [y_2]_j^\kappa \end{pmatrix} = 0, \kappa = 0,\dots,K; j = 0,\dots,J.$$

In order to prove the exponential stability of the difference initial-boundary value problem (4) - (7), we propose the following function as a candidate for the discrete Lyapunov function

$$V(\kappa\Delta) = V^\kappa \equiv V_1^\kappa + V_2^\kappa, U(\kappa\Delta) = U^\kappa \equiv U_1^\kappa + U_2^\kappa,$$

where

$$\begin{cases} V_1^\kappa \triangleq \Delta x \sum_{j=1}^{J} f_1(x_j)[(y_1)_j^\kappa]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right), \\ V_2^\kappa \triangleq \Delta x \sum_{j=0}^{J-1} f_2(x_j)[(y_2)_j^\kappa]^2 \exp\left(\frac{\mu}{(\lambda_2)_j} x_j\right), \end{cases} \tag{9}$$

$$\begin{cases} U_1^\kappa \triangleq \Delta x \sum_{j=1}^{J} f_1(x_j)[(z_1)_j^\kappa]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right), \\ U_2^\kappa \triangleq \Delta x \sum_{j=0}^{J-1} f_2(x_j)[(z_2)_j^\kappa]^2 \exp\left(\frac{\mu}{(\lambda_2)_j} x_j\right), \end{cases} \tag{10}$$

a parameter $\mu > 0$ and two functions $f_1 \in C^1([0,L];(0,+\infty))$, $f_2 \in C^1([0,L];(0,+\infty))$, are to be determined.

Lemma 2.1: *Let the conditions of Theorem 2.1 be satisfied. Then the following inequality holds:*

$$\frac{U_1^\kappa - V_1^\kappa}{\Delta t} \leq \sum_{j=1}^{J} f_1(x_j)(\lambda_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)\left\{[(y_1)_{j-1}^\kappa]^2 - [(y_1)_j^\kappa]^2\right\} \tag{11}$$

Proof: Let us denote by $(\rho_1)_j$ the Courant numbers $(\lambda_1)_j \frac{\Delta t}{\Delta x}$ for the first difference equation of system (4). Then the first difference equation of system (4) takes the form:

$$(z_1)_j^\kappa = (y_1)_j^\kappa - (\rho_1)_j[(y_1)_j^\kappa - (y_1)_{j-1}^\kappa], j = 1, \dots, J; \kappa = 0, \dots, K-1.$$

Taking into account this form of writing the first difference equation of system (4), we obtain the following expression for $\frac{U_1^\kappa - V_1^\kappa}{\Delta t}$,

$$\frac{U_1^\kappa - V_1^\kappa}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j)[(z_1)_j^\kappa]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)$$

$$- \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j)[(y_1)_j^\kappa]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)$$

$$= \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j)\left\{[(z_1)_j^\kappa]^2 - [(y_1)_j^\kappa]^2\right\} \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right).$$

In this equality, instead of $(z^1)_j^K$ we substitute its value from the first difference equation of system (4),

$$\frac{U_1^\kappa - V_1^\kappa}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j) \begin{Bmatrix} \{(y_1)_j^\kappa - (\rho_1)_j[(y_1)_j^\kappa - (y_1)_{j-1}^\kappa]\}^2 \\ -[(y_1)_j^\kappa]^2 \end{Bmatrix} \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right).$$

Let us simplify the expression inside the curly braces

$$Y_j = \langle (y_1)_j^{\kappa} - (\rho_1)_j [(y_1)_j^{\kappa} - (y_1)_{j-1}^{\kappa}]\rangle^2 - [(y_1)_j^{\kappa}]^2$$

$$= [(y_1)_j^{\kappa}]^2 - 2(\rho_1)_j (y_1)_j^{\kappa}[(y_1)_j^{\kappa} - (y_1)_{j-1}^{\kappa}]$$

$$+ [(\rho_1)_j]^2 \langle [(y_1)_j^{\kappa}]^2 - 2(y_1)_j^{\kappa}(y_1)_{j-1}^{\kappa} + [(y_1)_{j-1}^{\kappa}]^2\rangle - [(y_1)_j^{\kappa}]^2$$

$$= -2(\rho_1)_j [(y_1)_j^{\kappa}]^2 + 2(\rho_1)_j (y_1)_j^{\kappa}(y_1)_{j-1}^{\kappa}$$

$$+ [(\rho_1)_j]^2 [(y_1)_j^{\kappa}]^2 - 2[(\rho_1)_j]^2 (y_1)_j^{\kappa}(y_1)_{j-1}^{\kappa} + [(\rho_1)_j]^2 [(y_1)_{j-1}^{\kappa}]^2$$

According to the CFL condition, we have

$$(\rho_1)_j \leq \frac{\Delta t}{\Delta x} \max_{\substack{1 \leq i \leq 2 \\ 1 \leq j \leq J}} |(\lambda_i)_j| \leq 1$$

whence it follows that the inequality $1 - (\rho_1)_j > 0$. Consequently, the inequality $(\rho_1)_j (1 - (\rho_1)_j > 0$. Using the algebraic inequality $2ab \leq a^2 + b^2$, we will get

$$Y_j = -(\rho_1)_j [(y_1)_j^{\kappa}]^2 + [(\rho_1)_j]^2 [(y_1)_j^{\kappa}]^2 - (\rho_1)_j [(y_1)_j^{\kappa}]^2$$

$$+ (\rho_1)_j [(y_1)_j^{\kappa}]^2 - [(\rho_1)_j]^2 [(y_1)_j^{\kappa}]^2 + (\rho_1)_j [(y_1)_{j-1}^{\kappa}]^2$$

$$- [(\rho_1)_j]^2 [(y_1)_{j-1}^{\kappa}]^2 + [(\rho_1)_j]^2 [(y_1)_{j-1}^{\kappa}]^2$$

$$= (\rho_1)_j \{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\}.$$

Thus, for the expression Y_j from above, we have the estimate

$$Y_j \leq (\rho_1)_j \{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\}.$$

Then for the finite-difference relation $\frac{U_1^{\kappa} - V_1^{\kappa}}{\Delta t}$, the following inequality holds,

$$\frac{U_1^{\kappa} - V_1^{\kappa}}{\Delta t} = \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j) Y_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)$$

$$\leq \frac{\Delta x}{\Delta t} \sum_{j=1}^{J} f_1(x_j)(\rho_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\}$$

$$= \sum_{j=1}^{J} f_1(x_j)(\lambda_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\}$$

or

$$\frac{U_1^{\kappa} - V_1^{\kappa}}{\Delta t} \leq \sum_{j=1}^{J} f_1(x_j)(\lambda_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right)\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\}.$$

The proof of Lemma 2.1 is complete. This Lemma 2.1 implies the following Lemma 2.2.

Lemma 2.2: *Let the conditions of Theorem 2.1 be satisfied. Then the following inequality holds:*

$$\frac{U_1^{\kappa} - V_1^{\kappa}}{\Delta t} \le (g_1)_0 [(y_1)_0^{\kappa}]^2 - (g_1)_J [(y_1)_J^{\kappa}]^2$$

$$+ \sum_{j=1}^{J} [(f_1 \lambda_1)_{j-1} - (f_1 \lambda_1)_j] \exp\left(-\frac{\mu}{(\lambda_1)_{j-1}} x_{j-1}\right) [(y_1)_j^{\kappa}]^2$$

$$- \Delta x \mu \sum_{j=1}^{J} (f_1)_j [(y_1)_j^{\kappa}]^2 \left[\exp\left(-\frac{\mu}{\lambda_1} x\right)\right]_j.$$

(12)

Proof: Let us introduce the notation

$$(g_1)_j = f_1(x_j)(\lambda_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j} x_j\right).$$

Now by Lemma 2.1 with accuracy $O(\Delta x)$, we have the inequality

$$\frac{U_1^{\kappa} - V_1^{\kappa}}{\Delta t} \le \sum_{j=1}^{J} (g_1)_{j-1} \left\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\right\}.$$

We use the difference distinction formula

$$(g_1)_{j-1} \left\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\right\}$$

$$= \left\{(g_1)_{j-1} [(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j [(y_1)_j^{\kappa}]^2\right\} - [(g_1)_{j-1} - (g_1)_j][(y_1)_j^{\kappa}]^2.$$

Then, according to this formula for difference distinction, we have

$$\sum_{j=1}^{J} (g_1)_{j-1} \left\{[(y_1)_{j-1}^{\kappa}]^2 - [(y_1)_j^{\kappa}]^2\right\}$$

$$= \sum_{j=1}^{J} \left\langle \left\{(g_1)_{j-1} [(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j [(y_1)_j^{\kappa}]^2\right\} - [(g_1)_{j-1} - (g_1)_j][(y_1)_j^{\kappa}]^2 \right\rangle.$$

We will divide this sum into two parts and study each part separately,

$$\sum_{j=1}^{J} \left\langle \left\{(g_1)_{j-1} [(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j [(y_1)_j^{\kappa}]^2\right\} - [(g_1)_{j-1} - (g_1)_j][(y_1)_{j-1}^{\kappa}]^2 \right\rangle$$

$$= \sum_{j=1}^{J} \left\{(g_1)_{j-1} [(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j [(y_1)_j^{\kappa}]^2\right\} + \sum_{j=1}^{J} [(g_1)_{j-1} - (g_1)_j][(y_1)_j^{\kappa}]^2.$$

(13)

Summing up the first part of the right-hand side of this equality, we obtain

$$\sum_{j=1}^{J} \left\{(g_1)_{j-1} [(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j [(y_1)_j^{\kappa}]^2\right\} = (g_1)_0 [(y_1)_0^{\kappa}]^2 - (g_1)_J [(y_1)_J^{\kappa}]^2.$$

We pass to the transformation of the second part of the right side of equality (13)

$$\sum_{j=1}^{J} [(g_1)_{j-1} - (g_1)_j][(y_1)_j^{\kappa}]^2$$

$$= \sum_{j=1}^{J} \left\{\left[f_1 \lambda_1 \exp\left(-\frac{\mu}{\lambda_1} x\right)\right]_{j-1} - \left[f_1 \lambda_1 \exp\left(-\frac{\mu}{\lambda_1} x\right)\right]_j\right\} [(y_1)_j^{\kappa}]^2.$$

According to the formula for difference distinction of the product, we have

$$\left[f_1\lambda_1\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_{j-1} - \left[f_1\lambda_1\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j$$

$$= \left[(f_1\lambda_1)_{j-1} - (f_1\lambda_1)_j\right]\exp\left(-\frac{\mu}{(\lambda_1)_{j-1}}x_{j-1}\right)$$

$$+ (f_1\lambda_1)_j\left\{\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_{j-1} - \left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j\right\}.$$

Let us make some transformation of the second term of this equality,

$$\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_{j-1} - \left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j$$

$$= \left\{1 - \left[\exp\left(\frac{\mu}{\lambda_1}x\right)\right]_j\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_{j-1}\right\}\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j$$

$$= \left\{1 - \exp\frac{\mu[(x)_j(\lambda_1)_{j-1} - (x)_{j-1}(\lambda_1)_j]}{(\lambda_1)_{j-1}(\lambda_1)_j}\right\}\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j$$

$$-\Delta x\frac{\mu}{(\lambda_1)_j}\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j + O(\Delta x^2).$$

Then from (13) with accuracy we obtain the following equality

$$\sum_{j=1}^{J}\left\langle\left\{(g_1)_{j-1}[(y_1)_{j-1}^{\kappa}]^2 - (g_1)_j[(y_1)_j^{\kappa}]^2\right\} - [(g_1)_{j-1} - (g_1)_j][(y_1)_{j-1}^{\kappa}]^2\right\rangle$$

$$= (g_1)_0[(y_1)_0^{\kappa}]^2 - (g_1)_J[(y_1)_J^{\kappa}]^2 + \sum_{j=1}^{J}[(f_1\lambda_1)_j - (f_1\lambda_1)_{j-1}]\exp\left(-\frac{\mu}{(\lambda_1)_{j-1}}x_{j-1}\right)[(y_1)_j^{\kappa}]^2$$

$$-\Delta x\mu\sum_{j=1}^{J}(f_1)_j[(y_1)_j^{\kappa}]^2\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j.$$

Hence, this gives inequality (12).

Lemma 2.3: *Let the conditions of Theorem 2.1 be satisfied. Then the following inequality holds:*

$$\frac{U_2^{\kappa} - V_2^{\kappa}}{\Delta t} \leq \sum_{j=0}^{J-1}f_2(x_j)(\lambda_2)_j\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left\{[(y_2)_{j+1}^{\kappa}]^2 - [(y_2)_j^{\kappa}]^2\right\}.$$

The proof of Lemma 2.3 is similar to the proof of Lemma 2.1. For this reason, we omit the proofs of Lemma 2.3.

Lemma 2.4: *Let the conditions of Theorem 2.1 be satisfied. Then the following inequality holds:*

$$\frac{U_2^{\kappa} - V_2^{\kappa}}{\Delta t} \leq (g_2)_J[(y_2)_J^{\kappa}]^2 - (g_2)_0[(y_2)_0^{\kappa}]^2$$

$$-\sum_{j=0}^{J-1}[(f_2\lambda_2)_{j+1} - (f_2\lambda_2)_j]\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_{j+1}[(y_2)_{j+1}^{\kappa}]^2$$

$$-\Delta x\mu\sum_{j=0}^{J-1}(f_2)_j[(y_2)_{j+1}^{\kappa}]^2\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_j. \tag{14}$$

Here

$$(g_2)_j = f_2(x_j)(\lambda_2)_j \exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right).$$

The proof of Lemma 2.4 is similar to the proof of Lemma 2.2. For this reason, we omit the proofs of Lemma 2.4.

Lemma 2.5: *Let the conditions of Theorem 2.1 be satisfied. Then, with accuracy O(Δt), the following equality holds:*

$$\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t} = -2\Delta x \sum_{j=1}^J a_j f_1(x_j)(z_1)_j^\kappa (z_2)_j^\kappa. \tag{16}$$

Proof. For $\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t}$ we get the following expression

$$\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t} = \frac{\Delta x}{\Delta t}\sum_{j=1}^J f_1(x_j)[(y_1)_j^{\kappa+1}]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)$$

$$-\frac{\Delta x}{\Delta t}\sum_{j=1}^J f_1(x_j)[(z_1)_j^\kappa]^2 \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)$$

$$= \frac{\Delta x}{\Delta t}\sum_{j=1}^J f_1(x_j)\left\{[(y_1)_j^{\kappa+1}]^2 - [(z_1)_j^\kappa]^2\right\} \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right).$$

In this equality, instead of $[(y_1)_j^{\kappa+1}]^2$, we substitute its value from the first difference equation of system (5),

$$\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t} = \frac{\Delta x}{\Delta t}\sum_{j=1}^J f_1(x_j)\left\{\begin{matrix}\left((z_1)_j^\kappa - \Delta t a_j(z_2)_{j-1}^\kappa\right)^2 \\ -[(z_1)_j^\kappa]^2\end{matrix}\right\} \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)$$

Given the obvious equality

$$\left((z_1)_j^\kappa - \Delta t a_j(z_2)_{j-1}^\kappa\right)^2 - [(z_1)_j^\kappa]^2 = [(z_1)_j^\kappa]^2 - 2\Delta t a_j(z_1)_j^\kappa(z_2)_{j-1}^\kappa$$

$$-(\Delta t)^2(a_j)^2[(z_2)_{j-1}^\kappa]^2 - [(z_1)_j^\kappa]^2 = -2\Delta t a_j(z_1)_j^\kappa(z_2)_{j-1}^\kappa + O((\Delta t)^2)$$

for the expression $\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t}$, we obtain

$$\frac{V_1^{\kappa+1}-U_1^\kappa}{\Delta t} = \frac{\Delta x}{\Delta t}\sum_{j=1}^J f_1(x_j)\{-2\Delta t a_j(z_1)_j^\kappa(z_2)_{j-1}^\kappa + O((\Delta t)^2)\} \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)$$

$$= \Delta x \sum_{j=1}^J f_1(x_j)\{-2a_j(z_1)_j^\kappa(z_2)_{j-1}^\kappa + O(\Delta t)\} \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)$$

$$\approx -2\Delta x \sum_{j=1}^J a_j f_1(x_j)(z_1)_j^\kappa(z_2)_{j-1}^\kappa.$$

Thus, with accuracy $O(\Delta t)$, equality (16) is true for the expression $\frac{V_1^{\kappa+1}-U_1^{\kappa}}{\Delta t}$. Lemma 2.5 is proved. Similarly, one can prove the following lemmas.

Lemma 2.6: *Let the conditions of Theorem 2.1 be satisfied. Then, with accuracy $O(\Delta t)$, the following inequality holds:*

$$\frac{V_2^{\kappa+1}-U_2^{\kappa}}{\Delta t} = -2\Delta x \sum_{j=0}^{J-1} b_j f_2(x_j)(z_1)_{j+1}^{\kappa}(z_2)_j^{\kappa}.$$

(17)

Lemma 2.7: *Let the conditions of Theorem 2.1 be satisfied. Then, with accuracy $O(\Delta t+\Delta x)$, the following inequality holds:*

$$\frac{V_1^{\kappa+1}-V_1^{\kappa}}{\Delta t} \le (g_1)_0[(y_1)_0^{\kappa}]^2 - (g_1)_J\left[(y_1)_J^{\kappa}\right]^2$$

$$-\sum_{j=1}^{J}\left[(f_1\lambda_1)_{j-1} - (f_1\lambda_1)_j\right]\exp\left(-\frac{\mu}{(\lambda_1)_{j-1}}x_{j-1}\right)\left[(y_1)_j^{\kappa}\right]^2$$

$$-\Delta x\mu\sum_{j=1}^{J}(f_1)_j\left[(y_1)_j^{\kappa}\right]^2\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j$$

$$-2\Delta x\sum_{j=1}^{J}a_j f_1(x_j)(z_1)_j^{\kappa}(z_2)_{j-1}^{\kappa}.$$

(18)

Proof. Indeed, taking into account equalities (13) and (16), we have

$$\frac{V_1^{\kappa+1}-V_1^{\kappa}}{\Delta t} = \frac{V_1^{\kappa+1}-U_1^{\kappa}+U_1^{\kappa}-V_1^{\kappa}}{\Delta t} = \frac{V_1^{\kappa+1}-U_1^{\kappa}}{\Delta t} + \frac{U_1^{\kappa}-V_1^{\kappa}}{\Delta t}$$

$$\le (g_1)_0[(y_1)_0^{\kappa}]^2 - (g_1)_J\left[(y_1)_J^{\kappa}\right]^2$$

$$-\sum_{j=1}^{J}\left[(f_1\lambda_1)_{j-1} - (f_1\lambda_1)_j\right]\exp\left(-\frac{\mu}{(\lambda_1)_{j-1}}x_{j-1}\right)\left[(y_1)_j^{\kappa}\right]^2$$

$$-\Delta x\mu\sum_{j=1}^{J}(f_1)_j\left[(y_1)_j^{\kappa}\right]^2\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j - 2\Delta x\sum_{j=1}^{J}a_j f_1(x_j)(z_1)_j^{\kappa}(z_2)_{j-1}^{\kappa}.$$

This finishes the proof of the Lemma 2.7.

In a similar way we can prove the following lemmas.

Lemma 2.8: *Let the conditions of Theorem 2.1 be satisfied. Then, with accuracy $O(\Delta t+\Delta x)$, the following inequality holds:*

$$\frac{V_2^{\kappa+1}-V_2^{\kappa}}{\Delta t} \le (g_2)_J\left[(y_2)_J^{\kappa}\right]^2 - (g_2)_0[(y_2)_0^{\kappa}]^2$$

$$-\sum_{j=0}^{J-1}\left[(f_2\lambda_2)_{j+1} - (f_2\lambda_2)_j\right]\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_{j+1}\left[(y_2)_j^{\kappa}\right]^2$$

$$-\Delta x\mu\sum_{j=0}^{J-1}(f_2)_j\left[(y_2)_j^{\kappa}\right]^2\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_j$$

$$-2\Delta x\sum_{j=0}^{J-1}b_j f_2(x_j)(z_1)_{j+1}^{\kappa}(z_2)_j^{\kappa}.$$

(19)

Lemma 2.9: *Let the conditions of Theorem 2.1 be satisfied. Then, with accuracy $O(\Delta t + \Delta x)$, the following inequality holds*

$$\frac{V^{\kappa+1} - V^{\kappa}}{\Delta t} \leq (g_1)_0 [(y_1)_0^{\kappa}]^2 - (g_1)_J [(y_1)_J^{\kappa}]^2 + (g_2)_J [(y_2)_J^{\kappa}]^2 - (g_2)_0 [(y_2)_0^{\kappa}]^2$$

$$+ \sum_{j=1}^{J} [(f_1 \lambda_1)_j - (f_1 \lambda_1)_{j-1}] \exp\left(-\frac{\mu}{(\lambda_1)_{j-1}} x_{j-1}\right) [(y_1)_j^{\kappa}]^2$$

$$- \sum_{j=0}^{J-1} [(f_2 \lambda_2)_{j+1} - (f_2 \lambda_2)_j] \left[\exp\left(\frac{\mu}{\lambda_2} x\right)\right]_{j+1} [(y_2)_j^{\kappa}]^2$$

$$- \Delta x \mu \sum_{j=1}^{J} (f_1)_j [(y_1)_j^{\kappa}]^2 \left[\exp\left(-\frac{\mu}{\lambda_1} x\right)\right]_j$$

$$- \Delta x \mu \sum_{j=0}^{J-1} (f_2)_j [(y_2)_j^{\kappa}]^2 \left[\exp\left(\frac{\mu}{\lambda_2} x\right)\right]_j - 2\Delta x \sum_{j=1}^{J} a_j f_1(x_j)(z_1)_j^{\kappa}(z_2)_{j-1}^{\kappa}$$

$$- 2\Delta x \sum_{j=0}^{J-1} b_j f_2(x_j)(z_1)_{j+1}^{\kappa}(z_2)_j^{\kappa}. \tag{20}$$

Proof. Indeed, taking into account equalities (18) and (19), we have

$$\frac{V^{\kappa+1} - V^{\kappa}}{\Delta t} = \frac{(V_1^{\kappa+1} + V_2^{\kappa+1}) - (V_1^{\kappa} + V_2^{\kappa})}{\Delta t} = \frac{V_1^{\kappa+1} - V_1^{\kappa}}{\Delta t} + \frac{V_2^{\kappa+1} - V_2^{\kappa}}{\Delta t}$$

$$\leq (g_1)_0 [(y_1)_0^{\kappa}]^2 - (g_1)_J [(y_1)_J^{\kappa}]^2$$

$$+ \sum_{j=1}^{J} [(f_1 \lambda_1)_j - (f_1 \lambda_1)_{j-1}] \exp\left(-\frac{\mu}{(\lambda_1)_{j-1}} x_{j-1}\right) [(y_1)_j^{\kappa}]^2$$

$$- \Delta x \mu \sum_{j=1}^{J} (f_1)_j [(y_1)_j^{\kappa}]^2 \left[\exp\left(-\frac{\mu}{\lambda_1} x\right)\right]_j - 2\Delta x \sum_{j=1}^{J} a_j f_1(x_j)(z_1)_j^{\kappa}(z_2)_{j-1}^{\kappa}$$

$$(g_2)_J [(y_2)_J^{\kappa}]^2 - (g_2)_0 [(y_2)_0^{\kappa}]^2 - \sum_{j=0}^{J-1} [(f_2 \lambda_2)_{j+1} - (f_2 \lambda_2)_j] \left[\exp\left(\frac{\mu}{\lambda_2} x\right)\right]_{j+1} [(y_2)_j^{\kappa}]^2$$

$$- \Delta x \mu \sum_{j=0}^{J-1} (f_2)_j [(y_2)_j^{\kappa}]^2 \left[\exp\left(\frac{\mu}{\lambda_2} x\right)\right]_j - 2\Delta x \sum_{j=0}^{J-1} b_j f_2(x_j)(z_1)_{j+1}^{\kappa}(z_2)_j^{\kappa}$$

$$\leq (g_1)_0 [(y_1)_0^{\kappa}]^2 - (g_1)_J [(y_1)_J^{\kappa}]^2 + (g_2)_J [(y_2)_J^{\kappa}]^2 - (g_2)_0 [(y_2)_0^{\kappa}]^2$$

$$+ \sum_{j=1}^{J} [(f_1 \lambda_1)_j - (f_1 \lambda_1)_{j-1}] \exp\left(-\frac{\mu}{(\lambda_1)_{j-1}} x_{j-1}\right) [(y_1)_j^{\kappa}]^2$$

$$-\sum_{j=0}^{J-1}\left[(f_2\lambda_2)_{j+1}-(f_2\lambda_2)_j\right]\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_{j+1}\left[(y_2)_j^{\kappa}\right]^2$$

$$-\Delta x\mu\sum_{j=1}^{J}(f_1)_j\left[(y_1)_j^{\kappa}\right]^2\left[\exp\left(-\frac{\mu}{\lambda_1}x\right)\right]_j-\Delta x\mu\sum_{j=0}^{J-1}(f_2)_j\left[(y_2)_j^{\kappa}\right]^2\left[\exp\left(\frac{\mu}{\lambda_2}x\right)\right]_j$$

$$-2\Delta x\sum_{j=1}^{J}a_jf_1(x_j)(z_1)_j^{\kappa}(z_2)_{j-1}^{\kappa}-2\Delta x\sum_{j=0}^{J-1}b_jf_2(x_j)(z_1)_{j+1}^{\kappa}(z_2)_j^{\kappa}.$$

Lemma 2.9 is proved.

It is easy to verify that, with accuracy $O(\Delta t+\Delta x)$, the following inequality holds:

$$\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t}\le-\mu V^{\kappa}$$

$$-\left\{f_1(x_j)(\lambda_1)_j\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)\left[(y_1)_j^{\kappa}\right]^2-f_2(x_j)(\lambda_1)_j\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left[(y_2)_j^{\kappa}\right]^2\right\}_0^{J}$$

$$-\Delta x\sum_{j=1}^{J}\left|\begin{matrix}[-(f_1\lambda_1)_x]_j\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)\left[(y_1)_j^{\kappa}\right]^2+[(f_2\lambda_2)_x]_j\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left[(y_2)_{j-1}^{\kappa}\right]^2\\+2\left[a_jf_1(x_j)\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)+b_jf_2(x_j)\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\right](y_1)_j^{\kappa}(y_2)_{j-1}^{\kappa}\end{matrix}\right| \qquad (21)$$

Here we have used the notation and, with precision $O(\Delta x)$ equalities

1) $$\left\{f_1(x_j)(\lambda_1)_j\,exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)\left[(y_1)_j^{\kappa}\right]^2\right\}_0^{J}$$

$$=f_1(x_J)(\lambda_1)_J\exp\left(-\frac{\mu}{(\lambda_1)_J}x_J\right)\left[(y_1)_j^{\kappa}\right]^2-f_1(x_0)(\lambda_1)_0\left[(y_1)_j^{\kappa}\right]^2;$$

2) $$\left\{f_2(x_j)(\lambda_2)_j\,exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left[(y_2)_j^{\kappa}\right]^2\right\}_0^{J}$$

$$=f_2(x_J)(\lambda_2)_J\exp\left(\frac{\mu}{(\lambda_2)_J}x_J\right)\left[(y_2)_j^{\kappa}\right]^2-f_2(x_0)(\lambda_2)_0\left[(y_1)_j^{\kappa}\right]^2;$$

3) $f(x_j+\Delta x)=f(x_j)+\Delta xf'(x_j)+O(\Delta x^2)\approx f(x_j)+\Delta xf'(x_j),$

$$f'(x_j)=\frac{f(x_j+\Delta x)-f(x_j)}{\Delta x};$$

4) $f(x_j-\Delta x)=f(x_j)-\Delta xf'(x_j)+O(\Delta x^2)\approx f(x_j)-\Delta xf'(x_j),$

$$f'(x_j)=\frac{f(x_j)-f(x_j-\Delta x)}{\Delta x}.$$

Thus, we have the inequality

$$\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t} \leq -\mu V^{\kappa} + I_2 - I_1,\tag{22}$$

where

$$I_2^{\kappa} = -\left\{f_1(x_j)(\lambda_1)_j \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)\left[(y_1)_j^{\kappa}\right]^2 - f_2(x_j)(\lambda_1)_j \exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left[(y_2)_j^{\kappa}\right]^2\right\}_0^J$$

$$I_1^{\kappa} = \Delta x \sum_{j=1}^{J} \left| \begin{array}{l} \left[-(f_1\lambda_1)_x\right]_j \exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right)\left[(y_1)_j^{\kappa}\right]^2 + \left[(f_2\lambda_2)_x\right]_j \exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\left[(y_2)_{j-1}^{\kappa}\right]^2 \\ +2\left[a_j f_1(x_j)\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right) + b_j f_2(x_j)\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\right](y_1)_j^{\kappa}(y_2)_{j-1}^{\kappa} \end{array} \right|$$

In work [3], it is shown that there are functions f_1 and f_2 as well as a parameter μ_1 such that for all $\mu \in (0, \mu_1]$, the quadratic form of I_1 is positive for all $k = 0,...,K$. In addition, there is a $0 < \mu_2 < \mu_1$, such that for any $\mu \in (0, \mu_2]$ the quadratic form I_2^k is negative for all $\kappa = 0,...,K$. Therefore, from inequality (22), we obtain

$$\frac{V^{\kappa+1}-V^{\kappa}}{\Delta t} < -\mu V^{\kappa}.\tag{23}$$

Theorem 2.1: *Let $T > 0$ and the discrete Lyapunov function is determined using the formula (9-10). If the steps of the difference grid satisfy the CFL condition (8) and the parameters of the boundary conditions (2), b_0, b_1 obey the inequality*

$$b_0 \in -\left(\frac{g}{V^*(0)}, -\frac{V^*(0)}{H^*(0)}\right), b_1 \in \mathbb{R} \setminus \left[-\frac{g}{V^*(L)}, -\frac{V^*(L)}{H^*(L)}\right],$$

$$\tag{24}$$

then the numerical solution y_j^k of the difference initial-boundary value problem (1) - (4) is exponentially stable in the L^2-norm.

To prove Theorem 2.1, we use lemmas 2.1-2.9.

Proof of the theorem 2.1. Let the conditions of Theorem 2.1. be satisfied, then recursively applying inequality (23), we obtain the following inequality

$$V^{\kappa+1} < (1-\Delta t\mu)^{\kappa+1}V^0 \leq e^{-\mu\Delta t(\kappa+1)}V^0 = e^{-\mu t_{\kappa+1}}V^0, \kappa = 0,...,N-1.$$

We denote

$$C_1 = \min\left\{\min_{1\leq j\leq J} f_1(x_j)\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right), \min_{0\leq j\leq J-1} f_2(x_j)\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\right\},$$

$$C_2 = \max\left\{\max_{1\leq j\leq J} f_1(x_j)\exp\left(-\frac{\mu}{(\lambda_1)_j}x_j\right), \max_{0\leq j\leq J-1} f_2(x_j)\exp\left(\frac{\mu}{(\lambda_2)_j}x_j\right)\right\},$$

Then

$$C_1 \Delta x \left\{ \sum_{j=1}^{J} [(y_1)_j^\kappa]^2 + \sum_{j=0}^{J-1} [(y_2)_j^\kappa]^2 \right\} \leq V^\kappa \leq C_2 \Delta x \left\{ \sum_{j=1}^{J} [(y_1)_j^\kappa]^2 + \sum_{j=0}^{J-1} [(y_2)_j^\kappa]^2 \right\}, \kappa = 1, \ldots, N;$$

$$C_1 \Delta x \left\{ \sum_{j=1}^{J} [(y_1)_j^\kappa]^2 + \sum_{j=0}^{J-1} [(y_2)_j^\kappa]^2 \right\} \leq V^\kappa \leq e^{-\mu t_\kappa} V^0$$

$$\leq C_2 e^{-\mu t_\kappa} \left\{ \sum_{j=1}^{J} [(y_1)_j^0]^2 + \sum_{j=0}^{J-1} [(y_2)_j^0]^2 \right\}, \kappa = 1, \ldots, N.$$

Therefore

$$\Delta x \left\{ \sum_{j=1}^{J} [(y_1)_j^\kappa]^2 + \sum_{j=0}^{J-1} [(y_2)_j^\kappa]^2 \right\}$$

$$\leq C e^{-\mu t_\kappa} \Delta x \left\{ \sum_{j=1}^{J} [(y_{10})_j]^2 + \sum_{j=0}^{J-1} [(y_{20})_j]^2 \right\}, \kappa = 1, \ldots, N; C = C_2/C_1.$$

Thus, the last inequality means the exponential stability of the numerical solution of the y_j^k of the mixed problem is in the L^2-norm. And we have proved the theorem 2.1.

NUMERICAL EXAMPLE

We consider a pool of a prismatic open channel with a rectangular cross-section. As the values of the parameters of a numerical example, consider the following: channel length $L = 2000$ m, length step $\Delta x = 0.667$, time $t = 1$, time step $\Delta t = 0.002$, channel width $W = 70$ m, stream $Q = 70$ m³/sec, the stream rate of change $Q^* = 1$, and constant coefficient of friction $k = 0.001$.

Since we are considering runoff in river mode, the values H^* and V^* must correspond to the inequality $gH^* > V^{*2}$, and only in this case, $\lambda_1(x), \lambda_2(x)$ will take a positive value ($\lambda_1(x) > 0$, $\lambda_2(x) > 0$). In addition, we must not forget that $H^*(x)V^*(x) = Q^*$. Taking into account the above H^* and V^*, we have chosen as

$$V^*(x) = 0.01 \sin 2x + 1,$$

$$H^*(x) = \frac{1}{0.01 \sin 2x + 1}.$$

Accordingly, we take $\lambda_1(x), \lambda_2(x)$ as

$$\lambda_1 = \sqrt{\frac{9.8}{0.001 \sin 2x + 1} + 0.001 \sin 2x + 1},$$

$$\lambda_2 = \sqrt{\frac{9.8}{0.001 \sin 2x + 1} - (0.001 \sin 2x + 1)}.$$

We know that stationary states have three possible dynamics depending on the slope. In this numerical experiment, we consider the case when $gC = \frac{kV^{*2}}{H^*}$, which means that friction and tilt "compensate" each other. Then the bottom slope will have the form

$$C = \frac{k(0.001 \sin 2x + 1)^3}{9.8}.$$

Initial conditions

$$y_{10}(x) = \exp\left(\int_0^x \frac{-\frac{3(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1\right)}+k(0.001\sin 2x+1)^2-\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}}}{\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1}\,ds\right)$$

$$\times\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}\cos 2\pi z-\sin 2\pi x\right),$$

$$y_{20}(x) = \exp\left(\int_0^x \frac{\frac{3(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1\right)}+k(0.001\sin 2x+1)^2+\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}}}{\sqrt{\frac{9.8}{0.001\sin 2x+1}}-0.001\sin 2x+1}\,ds\right)$$

$$\times\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}\cos 2\pi z+\sin 2\pi x\right).$$

Also, we will be given

$$\gamma_1(x) = -\frac{3(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1\right)}+k(0.001\sin 2x+1)^2$$

$$-\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}},$$

$$\delta_1(x) = -\frac{(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1\right)}+k(0.001\sin 2x+1)^2$$

$$+\frac{k(0.001\sin 2x+1)^2}{2H*\sqrt{gH^*}},$$

$$\gamma_2(x) = \frac{(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}-0.001\sin 2x-1\right)}+k(0.001\sin 2x+1)^2-\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}},$$

$$\delta_2(x) = \frac{3(k(0.001\sin 2x+1)^3-9.8C)}{4\left(\sqrt{\frac{9.8}{0.001\sin 2x+1}}-0.001\sin 2x-1\right)}+k(0.001\sin 2x+1)^2$$

$$+\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}},$$

$$\varphi_1(x) = \exp\left(\int_0^x \frac{-\frac{3\left(k(0.001\sin 2x+1)^3-9.8C\right)}{4\left(\sqrt[4]{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1\right)}+k(0.001\sin 2x+1)^2-\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}}}{\sqrt{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1}\,ds\right),$$

$$\varphi_2(x) = \exp\left(\int_0^x \frac{\frac{3\left(k(0.001\sin 2x+1)^3-9.8C\right)}{4\sqrt[4]{\frac{9.8}{0.001\sin 2x+1}}+0.001\sin 2x+1}+k(0.001\sin 2x+1)^2+\frac{k(0.001\sin 2x+1)^3}{2\sqrt{\frac{9.8}{0.001\sin 2x+1}}}}{\sqrt{\frac{9.8}{0.001\sin 2x+1}}-0.001\sin 2x+1}\,ds\right),$$

$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)}.$$

Then, to determine the numerical solution (1) – (3), we use the difference splitting scheme for the lowest terms (4) – (7). According to Theorem 2.1, the conditions for exponential stability of the numerical solution take place in the fulfilment of the CFL conditions (8), as well as the parameters of the boundary conditions b_0, b_1 must satisfy the inequality (11). Simple calculations show that the CFL condition

$$\frac{\varDelta t}{\varDelta x}\max_{\substack{1\leq i\leq 2\\1\leq j\leq J}}|(\lambda_i)_j| = 0.012.$$

Obviously, the values $b_0 = -0.001$ and $b_1 = 0.001$ satisfy the inequality (11), it means

$$-0.001 \in (-9.8\times 10^4, -1\times 10^{-8}),$$
$$0.001 \in R\backslash(-4.667\times 10^3, -4.41\times 10^{-6}).$$

Let us give a graph of the L^2- norm of the numerical solution in Figure 1 which confirms the exponential stability of the numerical solution in the L^2- norm.
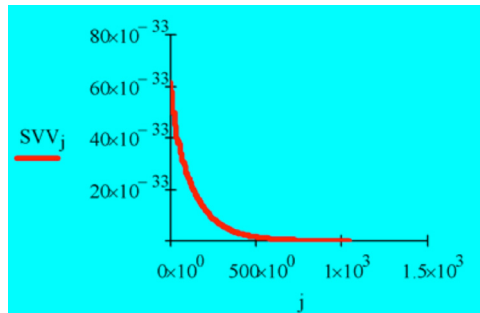

Figure 1: L^2- norm of the numerical solution

Now consider the case when the CFL condition is not satisfied

$$\frac{\varDelta t}{\varDelta x}\max_{\substack{1\leq i\leq 2\\1\leq j\leq J}}|(\lambda_i)_j| = 1.579$$

In this case, the L^2 - norm of the numerical solution increases with time, which means its instability (see Figure 2).
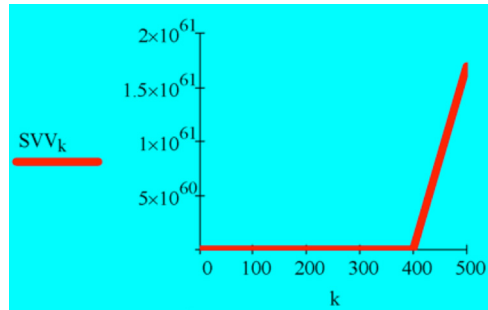

Figure 2: Instability of the numerical solution

Using the obtained numerical solution, the change in the height of the river flow at each moment Δt was calculated (see Figure 3).
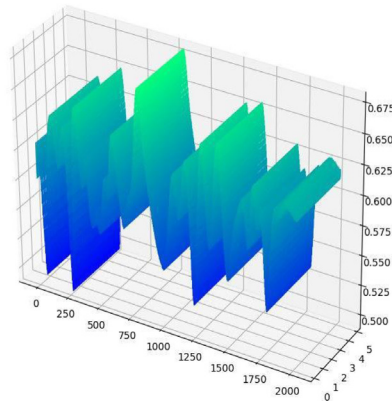

Figure 3: 3D model of flow height change $H(t,x)$

CONCLUSION

In this note, as conclusion, we have:

1. Constructed an explicit upwind difference splitting scheme in the lowest terms for the linear Saint-Venant equations in the general case;

2. Constructed a discrete analogue of the Lyapunov function for the numerical solution of the mixed problem for the system of linear Saint-Venant equations;

3. Obtained an a priori estimate for the discrete Lyapunov function;

4. Proved a theorem on the exponential stability of a numerical solution in the corresponding norm; and

5. Performed numerical calculations confirming the theoretical results of the exponential stability theorem.

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